E-45

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DEDICATION TO PUBLIC SERVICE *

BY

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Mr. President, members and guests of The Franklin Institute: Each year since 1944 a day has been set aside by The Franklin Institute to honor the memory of Charles Day who, in his time, was a devoted public servant, an engineer, a tactful, modest, and inspired leader in the world of large affairs.

Charles Day

His talents were many. Foremost among his achievements was pioneer work in the field of scientific management. He developed a system of cost accounting which was adopted, in its essentials, by the electric utility industry. During the first World War, he served on a number of missions for the United States government. He was a member of the Board of Managers and for several years a vice-president of The Franklin Institute.

Shortly before his death in 1931, Charles Day made a statement which typified his attitude throughout a brilliant and all-too-short career: "It is not the number of hours I have spent on this earth that matters," he said, "but only whether I have done my job."

Charles Day devoted many years of his life to the betterment of public service. He was eager to direct his talents and his professional skills to the solution of business problems of railroads and electric utilities. Many forward-looking contributions he made to sound business management left indelible marks on our way of life. The solid foundation of the electric light and power industry is a testimonial to the way in which Charles Day and his pioneer contemporaries met challenges twenty and more years ago.

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Birth of Public Service

In most fields of endeavor it is necessary to go back much farther than one or two score years to find "pioneers." But the electric industry is an exception. It is really not very old, as corporate life is measured; in fact, we might well say it is still a junior.

Less than seven decades have passed since the beginning of the electric utility business in the United States. Charles Day was a lad of three when Thomas A. Edison, in 1882, opened the first central station for the public sale of electricity—the little Pearl Street station in New York City. Water and gas services, railroads, the telegraph, and the telephone had already arrived.

Early Industry Pitfalls

As many in this room well know, the expansion of the electric utility industry to its present size and capabilities has not been a one-sided success story. There have been discouraging obstacles and pitfalls from the very outset.

The life of an industry is not unlike the life of an individual. Each passes through different phases—infancy, childhood, spirited youth, and mature vigor. Each has its own trials, troubles, and tribulations.

In its infancy and childhood, the electric utilities had the benefit of the inventive genius and technical skills of men like Thomas A. Edison, Charles J. Brush, and Philadelphia's native sons Elihu Thomson and Edwin J. Houston. The first dynamo tests ever made were carried out by this latter team in The Franklin Institute. Preceding these leaders by a full century was the man in whose memory this institution was founded—Benjamin Franklin—whose kite experiments in Philadelphia proved that lightning and man-made electricity are identical.

Great advances were made in the 1890's and early 1900's in the art of producing and distributing electricity. One of the greatest of these improvements was the substitution of the alternating current system of supply for the original direct current system.

The ever-present operating and engineering difficulties of those early days were further complicated by financing and organization problems. A little later came the need for a sound basis for utility regulation. These growing pains were not too serious, however, and they were absorbed by the lusty young giant without too much straining.

How Well Has the Job Been Done?

Applying Charles Day's test for a useful life on earth, I pose this question: How well has the electric utility industry done its job?

It is the duty and the responsibility of electric utilities to supply adequate and reliable light and power service; to meet the demands of all of the people, all of the time, at a fair price.

Meeting all needs means satisfying not only the normal year-in and year-out requirements of residential, business, and industrial customers, but also the soaring demands of war, as well as expanding needs during unforeseeable emergencies like the one that exists today.

To its great credit, the power industry is carrying the heavy load of the current defense effort without difficulty, even though these demands have been superimposed on a record-breaking peacetime consumption.

Power Never Too Little

This performance fully matches the industry's fine job during the past war, when all needs were met adequately and on time. There is no higher tribute to this war-time performance than the statement made at the end of World War II by J. A. Krug, head of the Office of War Utilities and later Secretary of the Interior. "Electric power," he said, "has never been too little nor too late."

Merits of Free Enterprise

Notwithstanding such noteworthy performances, one of the greatest challenges that face the electric utility industry today, and over the long range, is the need to make customers aware of the fact that the tax-paying, business-managed electric light and power companies of this country are producing an abundant supply of low-cost electricity; that they are able to do so because they have brought into being the most efficiently-designed and the most economically-operated electric generating and distributing system in the world.

This enviable position reflects the exercise of private initiative, and the application of unhampered energy, imagination, and inventive ability over the years. History shows there is no substitute for these ingredients of success, and that they flourish most effectively under free enterprise.

It is the duty of utility management to remind customers at every opportunity of the benefits they receive through the electric utility industry; also to make clear to them that the public at large has a great stake in the preservation of the investor-owned electric light and power industry, which makes these benefits possible.

Such a program will not only offset contrary claims, but it will overcome confused thinking which might otherwise lead to mistaken political decisions that would be contrary to the best interests of all.

Direct ownership of the electric utility industry is now in the hands of about 3,000,000 stockholders. Indirect ownership touches some 80,000,000 life insurance policy holders. This means that virtually every man, woman, and child in this country has an interest in the electric utility business. Doubtless a great majority of these individuals do not realize they have this stake in the business. One of the best things that could happen for the future of this country would be an even wider direct participation in the ownership of the electric light and power industry.

Fuel Generated Power

Many illustrations can be given of scientific management in action in the electric utility field. A good start would be to review the continuously increasing efficiencies which have been developed in the art of generating power.

Steam generation is the principal method used to produce electricity in America. This may not be realized by some, because of the wide publicity given water-power development in recent years. The fact of the matter is that hydro generation shows a slight decline, and today accounts for only 29 per cent of the total power generated in the United States.

This does not mean that hydro-electric generating capacity is unimportant, but it does reflect the degree to which the nation's growing power demands are being met by steam-generating stations.

Growing Fuel Economy

Fuels used to produce power in these steam plants are coal, oil, and gas. By far the most widely used is coal, which accounts for more than two-thirds of all fuel-produced electricity.

Expanding power requirements have increased the use of coal many, many times. Yet less coal per kilowatt-hour produced is required today than ever before.

Twenty years ago, the industry used one-and-one-half pounds of coal to generate a kilowatt-hour. Twenty years before that, four and one-half pounds were required. In 1892, the earliest year for which we have a record, it took eight pounds of coal to produce one kilowatt-hour of electricity. Today, the national average is slightly less than one and one-quarter pounds. New generating units on order and being installed will be even more efficient, for they will use only about two-thirds of a pound of coal for each kilowatt-hour of electricity produced.

This economical use of fuel to produce power reflects the coöperative efforts of the utilities and the manufacturers of the equipment to bring about the best results. The achievement highlights the efficiency of modern steam generating plants, and illustrates scientific management at its best. Not only have the utilities thus built up a high efficiency in fuel utilization, but in doing so they have contributed substantially to the conservation of one of the nation's most valuable natural resources.

Regional Power Pool

Another application of scientific management that has contributed greatly to the forward march of the electric utilities of the nation has been in the field of power interchange. Much has been published in recent years about "grid" systems and "power pools" formed by Federal agencies. Yet the fact is that Philadelphia Electric Company, Pennsylvania Power and Light Company, and Public Service Electric and Gas Company formed the first major system of electric power inter-connection over a quarter of a century ago.

The private utilities were alert to the advantages of this plan and that pioneering led to a great expansion of inter-company power connection. Today there are power pool areas in all sections of the country, extending even into Canada. More than 90 per cent of the nation's power generating capacity is interconnected in some degree. It is the exception to find any electric light and power company that has not pooled its generating facilities with its neighbor's, all in the interest of maintaining economical, adequate, and dependable service for customers.

Home Use of Electricity

The importance of these vast reservoirs of power is magnified by the ever-increasing demand for electricity.

Since 1939, the last pre-war year, the use of electricity in the average home has more than doubled! How does the average home manage to use twice as much electricity as in 1939? What is done with it? The size of the house has not doubled and there are still only three meals a day. Why, then, this large domestic use?

For the answer to these questions we need only look into a typical American home, from the electric laundry in the basement and the electric range and other equipment in the kitchen, to the television in the living room and the electric blanket in the bedroom. We find electric power serving the household in many different ways, like the slave who obeyed Aladdin's magic lamp.

Electrification of Farms

The advantages of electricity likewise have been brought to the farmers. Today 95 per cent of the farms in this country receive electric service, or have it available at the end of their lane.

Increasing Industrial Power Use

As remarkable as the increase in residential and rural uses of electricity has been, the growth in the use of electric power in industry has been equally striking. For example, industrial use of electricity has more than doubled in the past ten years. This is significant when it is considered that the substitution of electricity for human effort has enabled workers to manufacture more products in greater variety, more quickly, and more economically.

High Cost of Living

This reference to economics brings to mind the rising cost of living. This is a subject that is painfully familiar to all of us, particularly to the housewife, who has learned about it in her daily battle with the budget. To her, talk about a cost-of-living "index" or a "50-cent dollar" is a bit academic. She does know, however, that the higher prices she pays for the necessities of life are something to be considered. Actually, the latest official figure shows that the cost of living today is 185 per cent of the pre-war average.

Low Electric Rates

In view of these things, where can the housewife look for a bargain in living costs today? There is one sure place and that is in her electric bill, where her 1951 devalued dollar is buying about 40 per cent more electricity than her 1939 dollar bought.

Generous Use of Power

In America, the advantages of modern electric conveniences are not confined to any so-called well-to-do group, as is largely the case in many foreign countries, where electric service was developed and is operated under systems different from our own. On the contrary, the modern miracles of radio, television, electric refrigeration, and many others created by remarkable electric appliances are the commonplace comforts of the great mass of our citizens. In the Philadelphia area the benefits of television, one of the latest of these wonders, are enjoyed in thousands of homes.

V.
19

These far-reaching uses of electricity in America are reflected in world-wide statistics. Records for 1949, the latest available, show that the United States, with less than 7 per cent of the world population, produced 42½ per cent of all electric power generated in the entire world! Soviet Russia, the runner-up, is credited with 9.7 per cent, United Kingdom and Canada together, 12.7 per cent. No other nation produced as much as 5 per cent.

Downward Trend in Electric Rates

What explanation can be given for this seeming paradox of an ever-expanding demand being served at progressively lower rates?

Some suggest the influence of government power operations, which became especially active during the past two decades and have had widespread publicity. The fallacy of this explanation is that the average cost of electricity has been decreasing since the birth of the industry about seventy years ago.

Some give credit to the regulatory commissions. Careful supervision of rates by these bodies deserves much commendation. However, most of the commissions were established in comparatively recent years, while the steady drop in the cost of residential electricity has been going on since the earliest days. In 1882, when the first rate information was available, the average cost of electricity for residential use was 25 cents a kilowatt-hour. By 1900 it had dropped to 17 cents. By 1915, it was 8 cents. By 1930, it was 6 cents. Today it is about 3 cents—a reduction of 50 per cent in the past twenty years.

There must be some logical explanation for this constant decline through two World Wars, a great depression, and well into the present inflation! The facts are that the progressively lower rates charged for electricity were produced by a combination of many factors. First and foremost were availability and reliability of the service at fair rates, without which there could not have been successful promotion or new appliance development. It was these things, working together, and integrated by scientific management, which induced more and more people over the years to use more and more electricity. The explanation is as simple as that.

Service and Management at Work

This situation did not just happen. To gain the economic advantage of volume power production, the industry had to promote the use of electricity at attractive rates, and then be ready with prompt, efficient, and pleasant service to supply the greater volume of business thus produced. In other words, it was necessary to organize to handle more business, and then to attract additional business so as to keep generating transmission and distribution equipment operating most effectively.

This required an over-all sales activity that included not only the power companies' own organizations, but close coöperation with manufacturers, distributors, contractors, and dealers. Behind it all there had to be aggressive, imaginative supervision, technical "know-how," and ability to devise equipment and facilities which could attract the last drop of economic benefit from an expanding scale of operations.

And there had to be scientific management not only to correlate these interlocking activities, but to anticipate the pace of revenue growth and to keep the incentive of lower rates active in promoting more sales.

Unless industry planning had been well organized in all of these respects, periods of stagnation might have developed in the United States, just as they developed elsewhere. There has never been a period of arrested growth in the history of the electric industry in the United States except in the depression years of the early 1930's. Even then the declines were relatively slight. This is excellent evidence that the utilities were soundly conceived and have been and are efficiently operated.

Good Employee Relations

I have dwelt at length on the industry's responsibilities to the public because this is a fundamental and most important phase of the business.

However, the electric utility industry can continue to meet these responsibilities only so long as it can maintain satisfactory relationships with its employees and with its investors. Only by gauging its operations to pay a fair return to investors, and to provide good wages and good working conditions for employees can the industry continue to attract the kind of capital and skilled personnel needed to keep pace with the constantly growing demands for its service.

Relations with employees have been good. Utility employees, now numbering nearly 300,000, have steady, year-'round jobs, and over the past twelve years, their typical weekly wage has averaged from 18 to 33 per cent more than the wage level of employees in manufacturing. Other benefits have likewise kept pace.

Sound Investor Relations

In the realm of investor relations, the fact that the electric power business is dependent for its lifeblood, so to speak, on continued investor confidence, stands as a constant challenge to scientific management. A careful course must be charted and followed, because of the possibilities of encountering adverse factors, beyond utility control.

The capital investment in utility plant and equipment, equivalent, on the average, to four or five years' revenues, is in sharp contrast with capital requirements of manufacturing industries, as a group, where capital is turned over about once every 8½ months. The great need for capital in the electric utility business means that it is a "must" for electric light and power companies to enjoy high prestige with security owners and investors.

The unusually large investment required for electric utility plant facilities and equipment represents a sizable sum for each worker. For the utilities of the country the investment is equivalent to more than $56,000 per employee, which is eight times the average investment per employee in the nation's manufacturing industries.

It is interesting to note here that General Motors Corporation, last year, did half again as much business as all of the country's electric utilities, public and private put together, with an investment of only one-twentieth of that required for electric utilities!

Advancing Construction Costs

Everyone knows how rapidly construction costs have advanced since the war. Price increases of 100 per cent and over for material and labor are not uncommon. Despite these soaring costs, rising demands for power left utilities no alternative but to proceed with expansion programs.

In this emergency, they have carried on necessary construction work, and at a cost per unit of capacity added which is closely comparable with the long-term trend. This is another achievement of the highest merit and not generally appreciated and understood.

Here, once again, scientific management was operating at its best, for it was this essential component all along the line that produced the expert research, the improved designs, the careful planning, and the economical use of funds necessary to bring about so impressive a result.

Holding Company Act

A distinguishing characteristic of the electric utility industry has been its ability to accept criticism, to correct mistakes. Most of those present tonight remember the stormy times of the mid-thirties, when Congress passed The Public Utility Holding Company Act of 1935. It is human to recall only the abuses and consequent resentment which inspired the passage of this Act. Yet the holding company had its advantages, particularly in the early days of the industry, when operating companies were small and struggling. As a result, such companies often lacked public confidence and the benefit of broad scientific approach by management. It was only natural that many of them found it difficult to get necessary things done through their own efforts, and especially to arrange adequate financing. It was in these situations that the holding company was especially helpful.

Era of Friendship

The corporate integration policy followed by electric operating companies in recent years has been productive of much good.

The latest opinion polls show electric utilities today stand higher in the eyes of the public they serve than ever before. They have come a long way to win this confidence, and have earned it by the job they have done and are doing; and they are organized to continue to do this job in the future.

All of this points to mutual satisfaction and lasting friendship among customers, employees, and investors in the electric industry in the years ahead.

Unfair Competition

Moreover, they support the right of business-managed, tax-paying electric light and power companies to conduct the information program I mentioned earlier, and to speak out against the unfair competition from socialized and subsidized government ventures in public power.

Government power, under a system of tax exemption, interest-free financing, and other unjust preferences and subsidies at the expense of all tax-payers, today constitutes about one-fifth of the nation's capacity. Twenty years ago only 6 per cent of the nation's power was produced by the government. Soon or later the American people must consider, squarely on its merits, the question of how far this eroding tide of socialism will be permitted to go.

Three-fold Obligation

Summing up, may I suggest that the performance of the industry's job be appraised against its three-fold obligations: to the customer public; to the investors whose money it uses; and to its employees. All are of equal importance.

This three-fold obligation presents a never-ending challenge to the industry's resourcefulness and ingenuity. It calls for constant exercise of scientific management, skill, and alert business judgment. But most of all, it calls for a spirit of dedication to the ideals of public service over and above the call of duty in the average non-utility commercial enterprise.

Serving the Public

Those who have lived their lives in the electric utility industry know they are expected to give up some things which are the normal rights of others in unregulated business.

This is the principle one of our military leaders had in mind when he declared recently that "there are certain inhibitions you accept when you put on a uniform." We know, for example, our Company earnings will be restricted by law. We know we must conduct our affairs in a goldfish bowl of regulatory supervision. We know we cannot pick our customers. As soldiers of public service, we are always on call, and we are expected to deliver service day and night, rain or shine, Sundays and holidays, notwithstanding hurricanes, or disasters, or personal danger.

In the years ahead, I foresee that the work of the electric utility industry will be regarded as a profession. The difference between a profession and any other business vocation is not only specialized knowledge, but also a special sense of duty. The professional man feels indebted to his profession. Regardless of his own personal fortunes, he strives to be a credit to his profession and to keep faith with those who have gone before him.

Electric Industry Heritage

The electric utility industry has such a heritage, which has been handed down from the founding fathers of Edison's day. It has been upheld since then by those who have been busy with technical research and improvements, with promotion and expansion of operations, and with winning the public confidence and the just recognition which this great industry deserves.

In the memory of Charles Day, who had an important part in these developments, I salute them all! May we, who carry on, deserve the mantle of responsibility which has been passed on to us!

INTERFLECTANCE CALCULATIONS FOR VARIOUS LUMINAIRES

BY

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ABSTRACT

Previous interflectance tables have dealt principally with three basic types of lighting (Types I, II, and III). The present paper extends the scope of the interflection method by showing how it can be applied to rooms lighted by hanging luminaires. It is found that most cases arising in practice can be covered by six canonic distributions. The effect of mounting height is handled by means of a simple equation.

The integral equation for interflections of light has made possible the predetermination of pharosage and helios distributions in rooms (**1**).[3] On the basis of this theory, tables[4] have been prepared (**2**) to facilitate lighting design, and nomographs and slide rules have been devised for particular conditions (**3**).

There are three basic tables of interflectance—one for wall lighting, one for ceiling lighting, and one for floor lighting. The tables for ceiling lighting (Class IIa) are employed directly in the design of luminous ceilings of diffusing plastic (**4**). But most luminaires, as constructed today, do not give pure wall, ceiling, or floor lighting—they give a mixture of two or three of these basic types. Such hybrid lighting systems are calculated from the basic tables by the *method of superposition* (**5**).

To employ the superposition method, one must first determine the fractional parts of the luminaire output that reach the walls, ceiling, and floor. Obviously, these fractions depend not only on the light distribution from the luminaire but also on the positioning of the fixtures and on the shape of the room. We have customarily assumed that the spacing of the luminaires is equal to the mounting height, that the mounting height is $\frac{3}{4}$ of the ceiling height, and that the room is square (**6**). These assumptions are in reasonable agreement with traditional lighting design. Nevertheless, despite the direct application of the method by Coradeschi and Innes (**7**), it is often felt that present procedure in evaluating the fractions α_1, α_2, α_3 constitutes a weakness in the interflection method as applied to commercial fixtures.

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

[4] The published tables of interflectance f were computed for a principal surface that is above the floor by a distance equal to one-quarter of the ceiling height h. Consequently, all numerical values in the present paper are also for a principal surface at height $h/4$. An ordinary variation in table height has negligible effect on the interflectance; though values of f can be computed, if needed, for any height by use of the published equations (**2**).

Fortunately, in modern rooms with high reflectances, the average pharosage D_{av} is not greatly affected by the type of lighting. Even a change from direct to indirect lighting does not produce a very radical change in pharosage, while a variation of 10 per cent in an α may produce an immeasurably small change in D_{av}. In view of this non-critical nature of the light distribution, it is advisable to study the possibility of standardizing a relatively small number of distributions (say, from 6 to 10) which will cover the range ordinarily encountered in luminaire practice. Such a study is the purpose of the present paper.

SUPERPOSITION

The notation is as follows:

F_L = total pharos (lumen) from the lamps,
F_D = total pharos from the luminaires,
F_w = total pharos to principal surface (including interflections),
F_1 = pharos from luminaires directly to walls (no interflections),
F_2 = pharos from luminaires directly to ceiling,
F_3 = pharos from luminaires directly to floor,
F_{dir} = pharos from luminaires directly to principal surface,
D_{av} = average pharosage (lumen m^{-2}) incident on principal surface (including interflections),
S = area of floor (m^2),
S_1 = total area of walls (m^2),
$k_r = S_1/4S$ = domance of room (numeric),
$f = F_w/F_D$ = interflectance of room (numeric),
$f_1 = (F_w/F_D)_1$ = interflectance for wall lighting (Class Ib) only,
$f_2 = (F_w/F_D)_2$ = interflectance for ceiling lighting (Class IIb),
$f_3 = (F_w/F_D)_3$ = interflectance for floor lighting (Class IIIb),
$g = F_D/F_L$ = logance of luminaire (numeric),
$\alpha_1 = F_1/F_D$ = fraction of the luminaire pharos that is directly incident on walls (no interflections),
$\alpha_2 = F_2/F_D$ = fraction directly incident on ceiling,
$\alpha_3 = F_3/F_D$ = fraction directly incident on floor,
$\alpha_w = F_{dir}/F_D$ = fraction directly incident on principal surface,
s = spacing (m) between luminaires,
h = height (m) of room from floor to ceiling, and
h_m = mounting height (m) of luminaire above floor.

The average pharosage is

$$D_{av} = \frac{F_w}{S} = \left(\frac{F_w}{F_D}\right)\left(\frac{F_D}{F_L}\right)\frac{F_L}{S} = fg\frac{F_L}{S}. \qquad (1)$$

The problem of predetermining D_{av} in any room is thus reduced to a problem of finding the interflectance f. For the three basic types of

V.
19

lighting, values of f are read directly from tables. Values for the recommended reflectances:

$$\rho_1 = 0.50 \text{ (walls)},$$

$$\rho_2 = 0.80 \text{ (ceiling)},$$

$$\rho_3 = 0.30 \text{ (floor)}$$

are listed in Table I. These data will be used throughout the paper to show how the new method is applied to commercial luminaires. Exactly the same procedure can be applied when the room has other reflectances, the values of f_1, f_2, f_3 being taken from the published tables (2, 5).

TABLE I.—*Interflectance for* $\rho_1 = 0.50$, $\rho_2 = 0.80$, $\rho_3 = 0.30$.

k_r	Ib lighting f_1	IIb lighting f_2	IIIb lighting f_3*
0	0.5296	1.0526	0.3158
0.1	0.4611	0.9204	0.2509
0.2	0.4066	0.8105	0.2023
0.3	0.3622	0.7174	0.1654
0.4	0.3255	0.6374	0.1371
0.5	0.2948	0.5680	0.1151
0.6	0.2688	0.5073	0.0970
0.7	0.2464	0.4538	0.0827
0.8	0.2268	0.4063	0.0724
0.9	0.2102	0.3640	0.0642
1.0	0.1960	0.3265	0.0578

* Without direct light to the principal surface.

When the luminaires do not throw all their light to one surface, the principle of superposition is applied. The pharos from the luminaires is divided into three components, F_1, F_2, and F_3:

$$F_D = F_1 + F_2 + F_3. \tag{2}$$

Each component F_i produces on the principal surface a pharos $(F_w)_i$ which is independent of that produced by the other components. The separate values of pharos (including interflections) are

$$(F_w)_1 = f_1 F_1,$$

$$(F_w)_2 = f_2 F_2,$$

$$(F_w)_3 = f_3 F_3.$$

There is also a pharos F_{dir} directly incident on the principal surface.[5] Thus the total pharos incident on the principal surface, including all interflections, is

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[5] In previous applications of the superposition principle (Ref. 5), direct light has been included in f_3, and the assumption has been made that $\alpha_3 = \alpha_w$. The use of four terms in Eq. 3 instead of three seems to aid clear thinking, particularly when the mounting height of the luminaires is varied, and this modification has been employed throughout the present paper.

$$
\begin{aligned}
F_w &= (F_w)_1 + (F_w)_2 + (F_w)_3 + F_{dir} \\
&= f_1F_1 + f_2F_2 + f_3F_3 + F_{dir} \\
&= f_1\left(\frac{F_1}{F_D}\right) F_D + f_2\left(\frac{F_2}{F_D}\right) F_D + f_3\left(\frac{F_3}{F_D}\right) F_D + \left(\frac{F_{dir}}{F_D}\right) F_D
\end{aligned}
$$

or

$$f = f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + \alpha_w. \qquad (3)$$

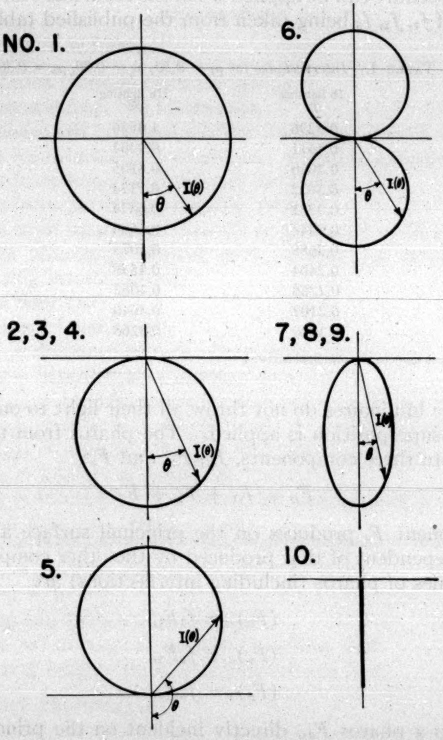

FIG. 1. Canonic distribution curves having simple mathematical forms.

The values of f_i are obtained from Table I. Thus if α_1, α_2, α_3, α_w are known, the resulting f for the complete luminaire is obtained from Eq. 3, and this value of f is used in Eq. 1 to calculate D_{av}.

V. 19

CANONIC DISTRIBUTIONS

The next step is to select a number of distribution curves which cover, in a fairly representative way, the range of curves obtained from actual luminaires. After several trials, six definite intensity-distribution curves were selected. Three mounting heights are employed with the two direct distributions, so the total number of arrangements is ten (Fig. 1). They are as follows:

1. Uniformly diffusing globes, $I(\theta) = I(0)$, $0 \leqq \theta \leqq \pi$, $h_m = \frac{3}{4}h$.
2. Direct lighting, broad distribution, $I(\theta) = I(0) \cos \theta$, $0 \leqq \theta \leqq \pi/2$, $h_m = h$.
3. Same as No. 2 except that $h_m = \frac{3}{4}h$.
4. Same as No. 2 except that $h_m = \frac{1}{2}h$.
5. Indirect lighting, $I(\theta) = -I(0) \cos \theta$, $\pi/2 \leqq \theta \leqq \pi$, $h_m = \frac{3}{4}h$.
6. Direct-indirect, $I(\theta) = I(0)|\cos \theta|$, $0 \leqq \theta \leqq \pi$, $h_m = \frac{3}{4}h$.
7. Direct lighting, narrow distribution, $I(\theta) = I(0) \cos^3 \theta$, $0 \leqq \theta \leqq \pi/2$, $h_m = h$.
8. Same as No. 7 except that $h_m = \frac{3}{4}h$.
9. Same as No. 7 except that $h_m = \frac{1}{2}h$.
10. Floor lighting (very narrow beam, no pharos to walls), any h_m.

Here all the distributions are symmetric about a vertical axis, and θ is the angle between a given ray and the vertical axis. Zero θ is taken toward the nadir. $I(\theta)$ is the luminous intensity in the direction θ. The mounting height is denoted by h_m and the ceiling height is h, both distances being measured from the floor.

THREE-DIMENSIONAL AND TWO-DIMENSIONAL PROBLEMS

Consider the calculation of α_3, the fractional part of the luminaire pharos that is directly incident on the floor. This quantity is usually determined as a function of k_r by assuming a fixed rule for luminaire spacing (such as $s = h_m$) and taking square rooms. The method, among its other disadvantages, tends to produce a discontinuous curve of α_3 *vs.* k_r. For example, if $s = h_m = h$, only one luminaire is used for all values of k_r greater than 1.00; but if $k_r = 0.9$, we must suddenly change to four luminaires, which increases the relative pharos to the walls, with consequent reduction in α_3. Similar discontinuities occur at the transition from 4 to 9 luminaires, 9 to 16, etc.

For this reason, it seemed advisable to modify the procedure so that an arduous summation is replaced by an integration. It was found that if the set of discrete luminaires is imagined to be replaced by a continuous distribution, each element of which has the same $I(\theta)/I(0)$ as the actual luminaire, the resulting α's are very nearly the same as those obtained by the older procedure. Figure 2 shows a typical comparison. The solid curve was obtained by integration, while the dotted curves were found by the older method. The lowest one of these dotted curves is the only one that applies to commercial practice. And this curve

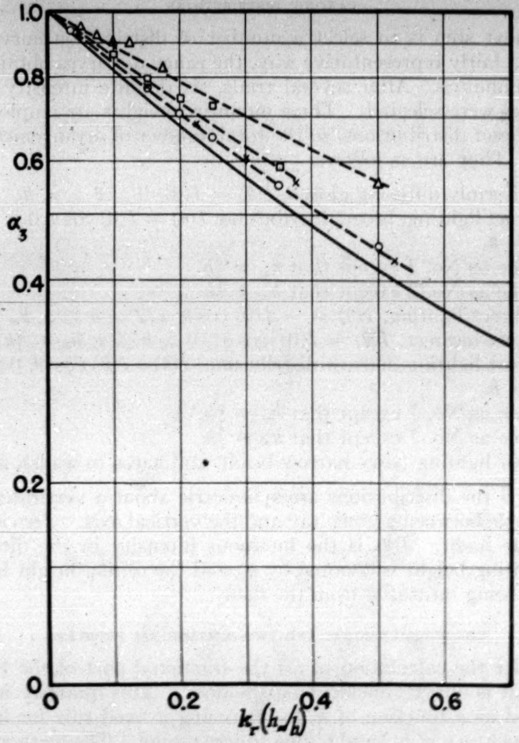

FIG. 2. Comparison of light distributions from discrete luminaires and from a fictitious continuous source, $I(\theta)/I(0) = \cos\theta$.

△. $s = 4h_m$;
⊡. $s = 2h_m$;
⊙. $s = h_m$;
×. $s = \frac{4}{3}h_m$;
— continuous distribution.

differs from the one obtained by integration by 6 per cent or less, which ordinarily causes a discrepancy in interflectance of considerably less than 6 per cent. On the basis of this and similar tests, we seem to be justified in replacing the actual discontinuous arrangement of luminaires by a fictitious continuous distribution.[6]

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[6] See also D. E. SPENCER, "Exact and Approximate Formulae for Illumination from Troffers," *Illum. Eng.*, Vol. 37, p. 596 (1942), where a fictitious continuous distribution of sources is found to produce a pharosage differing by less than 0.3 per cent from that produced by discrete luminaires.

The integrations are performed most conveniently if the rooms are considered to be infinitely long instead of square. The height and width are then related to domance by the equation,

$$k_r = h/2w, \tag{4}$$

and the problem becomes a two-dimensional one. The relations between the two-dimensional and the three-dimensional representations may be visualized from Fig. 3.

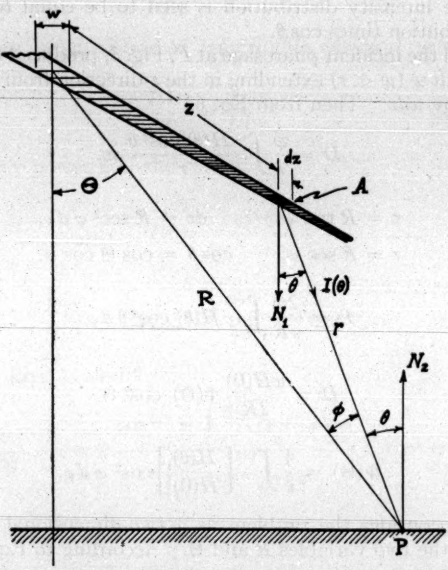

FIG. 3. Lighting from a continuous strip of width w and of infinite length.

Consider a small luminous area A on a fictitious horizontal surface at the level of the luminaires (Fig. 3). According to the familiar inverse-square relation, the incident pharosage at P is

$$D = \frac{I(\theta)}{r^2} \cos \theta, \tag{5}$$

where $I(\theta)$ is the luminous intensity in the direction from A to P. In terms of helios, the pharosage is

$$D = \frac{H(\theta)}{\pi} \frac{\cos^2 \theta}{r^2} A. \tag{6}$$

Thus we have a definite relation between the intensity distribution $I(\theta)$ of the luminaire and the helios distribution $H(\theta)$:

$$I(\theta) = \frac{H(\theta)}{\pi} \cos\theta \cdot A$$

or

$$\frac{I(\theta)}{I(0)} = \frac{H(\theta)}{H(0)} \cos\theta. \tag{7}$$

The relative intensity distribution is seen to be equal to the relative helios distribution times $\cos\theta$.

Now find the incident pharosage at P, Fig. 3, produced by a luminous strip of width w ($w \ll r$) extending in the z-direction from $-\infty$ to $+\infty$. Replace A by wdz. Then from Eq. 6,

$$D = \frac{w}{\pi} \int_{-\infty}^{\infty} \frac{H(\theta) \cos^2\theta}{r^2}\, dz.$$

But

$$z = R \tan\varphi, \qquad dz = R \sec^2\varphi\, d\varphi,$$

$$r = R \sec\varphi, \qquad \cos\theta = \cos\Theta \cos\varphi.$$

Thus,

$$D = \frac{2w}{\pi R} \int_0^{\pi/2} H(\theta) \cos^2\theta\, d\varphi$$

or

$$D = \frac{wH(0)}{2R} \Phi(\Theta) \cdot \cos^2\Theta \tag{8}$$

where

$$\Phi(\Theta) = \frac{4}{\pi} \int_0^{\pi/2} \left[\frac{H(\theta)}{H(0)} \right] \cos^2\varphi\, d\varphi. \tag{9}$$

We next consider the problem as a two-dimensional one, with P specified by the two variables R and Θ. According to Eq. 8,

$$D = \frac{wH(0)}{2R} \Phi(\Theta) \cdot \cos^2\Theta.$$

Or we may introduce a two-dimensional analogue $\bar{I}(\theta)$ of the usual intensity $I(\theta)$. In the two-dimensional case, the inverse square becomes an inverse first power, and in analogy with Eq. 5, one may write

$$D = \frac{\bar{I}(\Theta)}{R} \cos\Theta. \tag{10}$$

Comparison of Eqs. 8 and 10 shows that

$$\bar{I}(\Theta) = \frac{wH(0)}{2} \Phi(\Theta) \cdot \cos\Theta$$

or

$$\frac{\bar{I}(\Theta)}{\bar{I}(0)} = \frac{\Phi(\Theta)\cos\Theta}{\Phi(0)}. \tag{11}$$

Equation 11 allows the establishment of a theorem:

If the three-dimensional intensity distribution of the luminaires is

$$\frac{I(\theta)}{I(0)} = \cos^n \theta,$$

then the relative two-dimensional intensity is identical with the three-dimensional, or

$$\frac{\bar{I}(\Theta)}{\bar{I}(0)} = \frac{I(\theta)}{I(0)}. \tag{12}$$

Proof: Take any power n of the cosine:

$$\frac{I(\theta)}{I(0)} = \cos^n \theta.$$

Then, by Eq. 7,

$$\frac{H(\theta)}{H(0)} = \cos^{n-1}\theta = \cos^{n-1}\Theta \cos^{n-1}\varphi.$$

Therefore, by Eqs. 11 and 9,

$$\frac{\bar{I}(\Theta)}{\bar{I}(0)} = \cos\Theta \, \frac{\cos^{n-1}\Theta \displaystyle\int_0^{\pi/2} \cos^{n+1}\varphi \, d\varphi}{\displaystyle\int_0^{\pi/2} \cos^{n+1}\varphi \, d\varphi} = \cos^n \Theta,$$

which establishes Eq. 12. Values for a number of cases are listed in Table II.

TABLE II.

Distribution No.	$I(\theta)/I(0)$	$H(\theta)/H(0)$	$\Phi(\Theta)$
1	1	$\sec\theta$	$\frac{4}{\pi}\sec\Theta$
2, 3, 4	$\cos\theta$	1	1
	$\cos^2\theta$	$\cos\theta$	$\frac{8}{3\pi}\cos\Theta$
7, 8, 9	$\cos^3\theta$	$\cos^2\theta$	$\frac{3}{4}\cos^2\Theta$
	$\cos^4\theta$	$\cos^3\theta$	$\frac{32}{15\pi}\cos^3\Theta$
	$\cos^5\theta$	$\cos^4\theta$	$\frac{5}{8}\cos^4\Theta$

CALCULATION OF THE α'S

The next step is the calculation of the fractional parts of the luminaire output that reach the floor, walls, and ceiling. The room is assumed to be infinitely long in the direction perpendicular to the diagram, Fig. 4; and the fictitious luminous surface, which is imagined to replace the luminaires, is divided into strips of width $d\nu$, for each of which

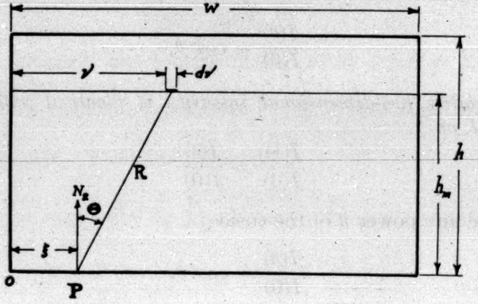

FIG. 4. Room of width w and length $l \rightarrow \infty$. The actual luminaires (at height h_m) are imagined to be replaced by a continuous source at this height.

$$\frac{\bar{I}(\Theta)}{\bar{I}(0)} = \frac{I(\theta)}{I(0)} = \frac{H(\theta)}{H(0)} \cos\theta.$$

For one strip, by Eq. 8,

$$dD_p = \frac{H(0)}{2R} \Phi(\Theta) \cos^2\Theta \, d\nu.$$

But $\cos^2\Theta = h_m^2/R^2$, $R^2 = h_m^2 + (\nu - \xi)^2$,

$$dD_p = \frac{h_m^2 H(0)}{2} \frac{\Phi(\Theta)\, d\nu}{[h_m^2 + (\nu - \xi)^2]^{3/2}}$$

and the total pharosage at P is

$$D_p = \frac{H(0)}{2} \int_0^w \frac{h_m^2 \Phi(\Theta)\, d\nu}{[h_m^2 + (\nu - \xi)^2]^{3/2}}. \tag{13}$$

The total pharos to the floor (for length l perpendicular to the diagram) is

$$F_3 = l \int_0^w D_p \, d\xi = \frac{lH(0)}{2} \int_0^w \int_0^w \frac{h_m^2 \Phi(\Theta)\, d\nu\, d\xi}{[h_m^2 + (\nu - \xi)^2]^{3/2}},$$

while the pharos from the source is

$$F_D = \frac{lH(0)}{2} \lim_{h_m \to 0} \int_0^w \int_0^w \frac{h_m^2 \Phi(\Theta)\, d\nu\, d\xi}{[h_m^2 + (\nu - \xi)^2]^{3/2}}.$$

Thus the fractional part of the direct light reaching the floor is

$$\alpha_3 = \frac{F_3}{F_D} = \frac{\displaystyle\int_0^w \int_0^w \frac{h_m{}^2\Phi(\Theta)\,d\nu\,d\xi}{[h_m{}^2 + (\nu - \xi)^2]^{3/2}}}{\displaystyle\lim_{h_m \to 0} \int_0^w \int_0^w \frac{h_m{}^2\Phi(\Theta)\,d\nu\,d\xi}{[h_m{}^2 + (\nu - \xi)^2]^{3/2}}}. \tag{14}$$

For example, if $I(\theta)/I(0) = \cos\theta$ (which represents the distributions from a large variety of commercial luminaires) then from Table II,

$$\Phi(\Theta) = 1.$$

Let $(\nu - \xi) = x$. Then

$$\int_0^w \frac{h_m{}^2\Phi(\Theta)d\nu}{[h_m{}^2 + (\nu - \xi)^2]^{3/2}} = \int_{-\xi}^{w-\xi} \frac{h_m{}^2\,dx}{[h_m{}^2 + x^2]^{3/2}} = \frac{(w - \xi)}{\sqrt{h_m{}^2 + (w - \xi)^2}} + \frac{\xi}{\sqrt{h_m{}^2 + \theta^2}}.$$

Also,

$$\int_0^w \left[\frac{w - \xi}{\sqrt{h_m{}^2 + (w - \xi)^2}} + \frac{\xi}{\sqrt{h_m{}^2 + \xi^2}}\right] d\xi = 2[\sqrt{h_m{}^2 + w^2} - h_m],$$

so

$$\alpha_3 = \frac{2[\sqrt{h_m{}^2 + w^2} - h_m]}{2w} = 2\left[\sqrt{k_r{}^2\left(\frac{h_m}{h}\right)^2 + \frac{1}{4}} - k_r\left(\frac{h_m}{h}\right)\right]. \tag{15}$$

Other formulas for α_3 are obtained in a similar manner and are given in Table III. The fraction α_2 for the ceiling is obtained in exactly the

TABLE III.—*Equations for α_2, α_3, and α_w.*

In all cases, $k_r = \frac{S_1}{4S}$ and $\alpha_1 = 1 - \alpha_2 - \alpha_3$.

No.	Distribution	h_m/h	α
1.	$I(\theta)/I(0) = 1$	$\frac{3}{4}$	$\alpha_2 = \frac{1}{\pi}\left[\tan^{-1}(2/k_r) - \frac{k_r}{4}\ln(1 + 4/k_r{}^2)\right]$ $\alpha_3 = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{2}{3k_r}\right) - \frac{3k_r}{4}\ln\left(1 + \frac{4}{9k_r{}^2}\right)\right]$ $\alpha_w = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1}{k_r}\right) - \frac{k_r}{2}\ln(1 + 1/k_r{}^2)\right]$
2.	$I(\theta)/I(0) = \cos\theta.$	1	$\alpha_2 = 0$ $\alpha_3 = \sqrt{4k_r{}^2 + 1} - 2k_r.$ $\alpha_w = \sqrt{\left(\frac{3k_r}{2}\right)^2 + 1} - \frac{3}{2}k_r$
3.	$I(\theta)/I(0) = \cos\theta.$	$\frac{3}{4}$	$\alpha_2 = 0$ $\alpha_3 = \sqrt{\left(\frac{3k_r}{2}\right)^2 + 1} - \frac{3}{2}k_r$ $\alpha_w = \sqrt{k_r{}^2 + 1} - k_r$
4.	$I(\theta)/I(0) = \cos\theta.$	$\frac{1}{2}$	$\alpha_2 = 0$ $\alpha_3 = \sqrt{k_r{}^2 + 1} - k_r$ $\alpha_w = \sqrt{(k_r/2)^2 + 1} - k_r/2$

TABLE III.—*Continued.*

5.	$I(\theta)/I(0) = -\cos\theta.$	$\frac{3}{4}$	$\begin{cases} \alpha_2 = \sqrt{(k_r/2)^2 + 1} - k_r/2 \\ \alpha_3 = 0 \\ \alpha_w = 0 \end{cases}$
6.	$I(\theta)/I(0) = \lvert\cos\theta\rvert.$	$\frac{3}{4}$	$\begin{cases} \alpha_2 = \frac{1}{2}[\sqrt{(k_r/2)^2 + 1} - k_r/2] \\ \alpha_3 = \frac{1}{2}\left[\sqrt{\left(\frac{3k_r}{2}\right)^2 + 1} - \frac{3}{2}k_r\right] \\ \alpha_w = \frac{1}{2}[\sqrt{k_r^2 + 1} - k_r] \end{cases}$
7.	$I(\theta)/I(0) = \cos^3\theta.$	1	$\begin{cases} \alpha_2 = 0 \\ \alpha_3 = 2\sqrt{k_r^2 + \frac{1}{4}} - \frac{k_r^2}{\sqrt{k_r^2 + \frac{1}{4}}} - k_r \\ \alpha_w = \sqrt{\left(\frac{3k_r}{2}\right)^2 + 1} - \frac{(3k_r/2)^2}{2\sqrt{(3k_r/2)^2 + 1}} - \frac{3k_r}{4} \end{cases}$
8.	$I(\theta)/I(0) = \cos^3\theta.$	$\frac{3}{4}$	$\begin{cases} \alpha_2 = 0 \\ \alpha_3 = \sqrt{\left(\frac{3k_r}{2}\right)^2 + 1} - \frac{(3k_r/2)^2}{2\sqrt{(3k_r/2)^2 + 1}} - \frac{3k_r}{4} \\ \alpha_w = \sqrt{k_r^2 + 1} - \frac{k_r^2}{2\sqrt{k_r^2 + 1}} - \frac{k_r}{2} \end{cases}$
9.	$I(\theta)/I(0) = \cos^3\theta.$	$\frac{1}{2}$	$\begin{cases} \alpha_2 = 0 \\ \alpha_3 = \sqrt{k_r^2 + 1} - \frac{k_r^2}{2\sqrt{k_r^2 + 1}} - \frac{k_r}{2} \\ \alpha_w = \sqrt{\left(\frac{k_r}{2}\right)^2 + 1} - \frac{(k_r/2)^2}{2\sqrt{(k_r/2)^2 + 1}} - \frac{k_r}{4} \end{cases}$
10.		Any.	$\begin{cases} \alpha_2 = 0 \\ \alpha_3 = 1 \\ \alpha_w = 1 \end{cases}$

same way except that in place of h_m we use $(h - h_m)$. Since the sum of α_1, α_2, and α_3 is always equal to unity, α_1 is obtained by subtraction:

$$\alpha_1 = 1 - \alpha_2 - \alpha_3. \tag{16}$$

The fraction α_w is a special case of α_3 in which $(h_m - h/4)$ replaces h_m. Calculated values of the α's are listed in Table IV.

TABLE IV.—*Distribution of Light from Luminaires.*

k_r	α_1	α_2	α_3	α_w
		No. 1		
0	0	0.5000	0.5000	0.5000
0.1	0.2021	0.4364	0.3615	0.3948
0.2	0.3174	0.3948	0.2878	0.3335
0.3	0.4007	0.3615	0.2378	0.2881
0.4	0.4654	0.3335	0.2011	0.2528

TABLE IV.—*Continued.*

k_r	α_1	α_2	α_3	α_w
0.5	0.5175	0.3093	0.1732	0.2243
0.6	0.5604	0.2881	0.1515	0.2011
0.7	0.5963	0.2694	0.1343	0.1817
0.8	0.6268	0.2528	0.1204	0.1654
0.9	0.6533	0.2378	0.1089	0.1515
1.0	0.6763	0.2243	0.09938	0.1397
		No. 2, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.1802		0.8198	0.8612
0.2	0.3230		0.6770	0.7440
0.3	0.4338		0.5662	0.6466
0.4	0.5194		0.4806	0.5662
0.5	0.5858		0.4142	0.4999
0.6	0.6380		0.3620	0.4454
0.7	0.6795		0.3205	0.4000
0.8	0.7132		0.2868	0.3620
0.9	0.7409		0.2591	0.3300
1.0	0.7639		0.2361	0.3028
		No. 3, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.1388		0.8612	0.9050
0.2	0.2560		0.7440	0.8198
0.3	0.3534		0.6466	0.7440
0.4	0.4338		0.5662	0.6770
0.5	0.5001		0.4999	0.6180
0.6	0.5546		0.4454	0.5662
0.7	0.6000		0.4000	0.5207
0.8	0.6380		0.3620	0.4806
0.9	0.6700		0.3300	0.4454
1.0	0.6972		0.3028	0.4142
		No. 4, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.0950		0.9050	0.9512
0.2	0.1802		0.8198	0.9050
0.3	0.2560		0.7440	0.8612
0.4	0.3230		0.6770	0.8198
0.5	0.3820		0.6180	0.7808
0.6	0.4338		0.5662	0.7440
0.7	0.4793		0.5207	0.7095
0.8	0.5194		0.4806	0.6770
0.9	0.5546		0.4454	0.6466
1.0	0.5858		0.4142	0.6180
		No. 5, $\alpha_3 = 0$, $\alpha_w = 0$		
0	0	1.0000		
0.1	0.0488	0.9512		
0.2	0.0950	0.9050		
0.3	0.1388	0.8612		
0.4	0.1802	0.8198		
0.5	0.2192	0.7808		
0.6	0.2560	0.7440		
0.7	0.2905	0.7095		

TABLE IV.—*Continued.*

k_r	α_1	α_2	α_3	α_w
0.8	0.3230	0.6770		
0.9	0.3534	0.6466		
1.0	0.3820	0.6180		
		No. 6		
0	0	0.5000	0.5000	0.5000
0.1	0.0938	0.4756	0.4306	0.4525
0.2	0.1755	0.4525	0.3720	0.4099
0.3	0.2461	0.4306	0.3233	0.3720
0.4	0.3070	0.4099	0.2831	0.3385
0.5	0.3597	0.3904	0.2499	0.3090
0.6	0.4053	0.3720	0.2227	0.2831
0.7	0.4452	0.3548	0.2000	0.2604
0.8	0.4805	0.3385	0.1810	0.2403
0.9	0.5117	0.3233	0.1650	0.2227
1.0	0.5396	0.3090	0.1514	0.2071
		No. 7, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.0998		0.9002	0.9251
0.2	0.1973		0.8027	0.8509
0.3	0.2883		0.7117	0.7793
0.4	0.3693		0.6307	0.7118
0.5	0.4394		0.5606	0.6499
0.6	0.4989		0.5011	0.5943
0.7	0.5492		0.4508	0.5447
0.8	0.5916		0.4084	0.5011
0.9	0.6275		0.3725	0.4626
1.0	0.6585		0.3415	0.4288
		No. 8, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.0749		0.9251	0.9500
0.2	0.1491		0.8509	0.9002
0.3	0.2207		0.7793	0.8509
0.4	0.2882		0.7118	0.8027
0.5	0.3501		0.6499	0.7562
0.6	0.4057		0.5943	0.7119
0.7	0.4553		0.5447	0.6700
0.8	0.4989		0.5011	0.6307
0.9	0.5374		0.4626	0.5943
1.0	0.5712		0.4288	0.5607
		No. 9, $\alpha_2 = 0$		
0	0		1.0000	1.0000
0.1	0.0500		0.9500	0.9750
0.2	0.0998		0.9002	0.9500
0.3	0.1491		0.8509	0.9251
0.4	0.1973		0.8027	0.9002
0.5	0.2438		0.7562	0.8755
0.6	0.2881		0.7119	0.8509
0.7	0.3300		0.6700	0.8266
0.8	0.3693		0.6307	0.8027
0.9	0.4057		0.5943	0.7792
1.0	0.4393		0.5607	0.7562

INTERFLECTANCE

After the α's are determined, the calculation of the interflectance f is a simple matter of superposition. The values of Table V were computed in this way, using Eq. 3. That the selected distributions (Fig. 1) result in curves that are fairly well spaced is indicated by Fig. 5. These curves are all for a mounting height equal to $\frac{3}{4}$ of the ceiling height.

TABLE V.—*Values of Interflectance f.*
(for $\rho_1 = 0.50$, $\rho_2 = 0.70$, $\rho_3 = 0.30$)

k_r	No. 1	2	3	4	5	6	7	8	9	10
0	1.1842	1.3158	1.3158	1.3158	1.0526	1.1842	1.3158	1.3158	1.3158	1.3158
0.1	0.9804	1.1500	1.1851	1.2221	0.8980	1.0416	1.1970	1.2166	1.2365	1.2509
0.2	0.8402	1.0123	1.0744	1.1441	0.7721	0.9233	1.0935	1.1330	1.1727	1.2023
0.3	0.7319	0.8974	0.9790	1.0770	0.6681	0.8236	1.0014	1.0597	1.1198	1.1654
0.4	0.6445	0.8012	0.8958	1.0178	0.5812	0.7385	0.9186	0.9941	1.0745	1.1371
0.5	0.5725	0.7203	0.8229	0.9645	0.5081	0.6655	0.8439	0.9342	1.0344	1.1151
0.6	0.5125	0.6519	0.7585	0.9156	0.4462	0.6024	0.7771	0.8786	0.9975	1.0970
0.7	0.4620	0.5939	0.7016	0.8707	0.3936	0.5476	0.7174	0.8272	0.9634	1.0827
0.8	0.4188	0.5445	0.6515	0.8296	0.3484	0.5000	0.6648	0.7801	0.9320	1.0724
0.9	0.3821	0.5024	0.6074	0.7918	0.3096	0.4585	0.6183	0.7370	0.9026	1.0642
1.0	0.3512	0.4662	0.5684	0.7568	0.2765	0.4225	0.5776	0.6973	0.8747	1.0578

The effect of mounting height is shown in Fig. 6. The curves indicate that the data can be represented by the equation,

$$f = 1.3158(10)^{-\zeta\frac{h_m}{h}} \tag{17}$$

where ζ represents the slope of the curve. Figure 6 is plotted for the cosine-cubed distribution (No. 8), but a similar graph is obtainable for the cosine distribution (No. 3). Numbers 5 and 6 also give results that can be represented by straight lines, though there is no common point at $h_m = 0$ as in Fig. 6.

Probably the most convenient way of handling mounting heights that are not included in Table V is to multiply the interflectance $f_{0.75}$ at $h_m/h = 0.75$ by a factor:

$$f/f_{0.75} = (10)^{\left(0.75-\frac{h_m}{h}\right)\zeta} \tag{18}$$

which may be written

$$\log\,(f/f_{0.75}) = \left(0.75 - \frac{h_m}{h}\right)\zeta. \tag{18a}$$

Values of ζ are listed in Table VI.

For example, suppose that the interflectance is needed for $k_r = 1.0$ and for Luminaire No. 3 but at a mounting height $h_m = 0.60h$. Then Eq. 18 becomes

$$f/f_{0.75} = (10)^{(0.75-0.60)0.495} = 1.186.$$

From Table V, $f_{0.75}$ for No. 3 is 0.5684, so at $h_m/h = 0.60$

$$f = 1.186\ (0.5684) = 0.6741.$$

TABLE VI.—*Values of* ζ.

k_r	For $\cos\theta$ (No. 3)	For $\cos^3\theta$ (No. 8)	No. 6	Indirect (No. 5)
0	0.000	0.000	0.000	0.000
0.1	0.060	0.042	0.018	−0.040
0.2	0.117	0.083	0.036	−0.078
0.3	0.171	0.124	0.054	−0.114
0.4	0.221	0.163	0.074	−0.147
0.5	0.268	0.200	0.096	−0.176
0.6	0.314	0.236	0.118	−0.200
0.7	0.360	0.270	0.141	−0.218
0.8	0.405	0.304	0.165	−0.232
0.9	0.450	0.337	0.189	−0.244
1.0	0.495	0.370	0.214	−0.255

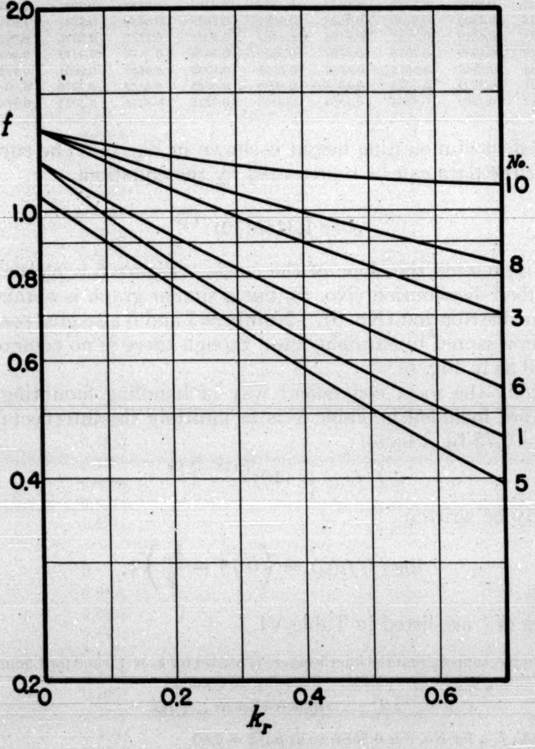

FIG. 5. Calculated interflectance for the six canonic distributions of Fig. 1. In all cases $h_m/h = 0.75$.

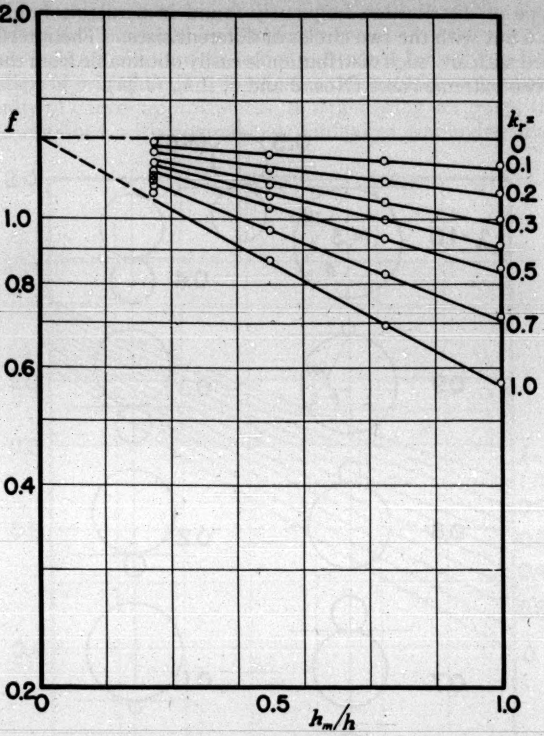

FIG. 6. Effect of mounting height of the luminaires for $I(\theta)/I(0) = \cos^3\theta$.

OTHER DISTRIBUTIONS

Figure 5 gives curves of interflectance *vs.* domance for six characteristic intensity distributions. In the range of 0 to 0.7 in k_r, the maximum deviation between successive curves is 30 per cent. For many practical cases, these curves suffice. Much industrial lighting is done with the RLM reflector, whose distribution curve is closely approximated by the cosine distribution (No. 3). The same distribution applies to many of the direct luminaires employed in offices. The $\cos^3\theta$ distribution is a close approximation to most "high-bay" units and other directive luminaires. All commercial indirect luminaires provide distributions very much like No. 5, while many with open top and bottom are like No. 6.

A type of distribution frequently found in available luminaires is like No. 6 but with the two circles of different sizes. The interflectance associated with any such distribution is easily obtainable from the results for the two extreme cases (Nos. 3 and 5, if $h_m = \frac{3}{4}h$).

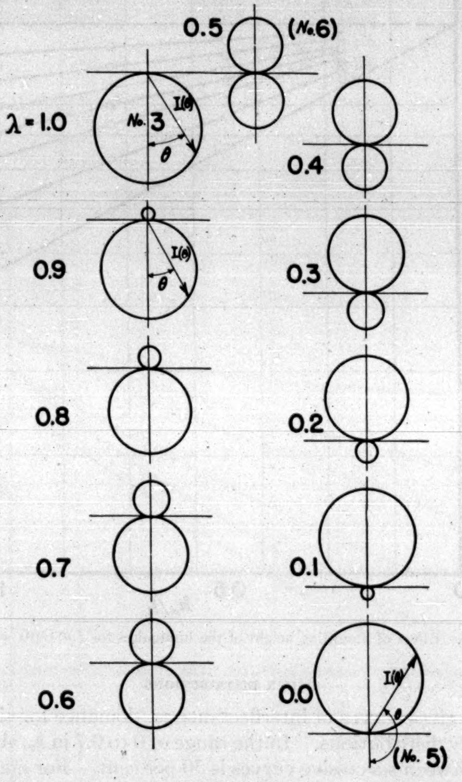

FIG. 7. Intermediate intensity distributions between No. 3 and No. 5, obtained by varying the relative amounts of light thrown upward and downward.

Introduce a new parameter λ, defined by the equation,

$$\lambda = \frac{I(0)}{I(0) + I(\pi)}. \tag{19}$$

Then for the complete luminaire,

$$f = \lambda f_L + (1 - \lambda) f_U, \tag{20}$$

where f_L is the interflectance produced by distribution No. 3 and f_U is the interflectance produced by No. 5.

Figure 7 shows eleven distributions obtained by varying λ from 1.0 to 0 in steps of 0.1, while Fig. 8 indicates the resulting curves of f *vs.* k_r. This family of curves approximates, to engineering accuracy, the interflectance obtained with commercial luminaires employing fluorescent lamps and ranging from those with opaque or translucent bottoms to the louvered and the RLM types. Still other styles can be approximated by combinations of the six canonical distributions.

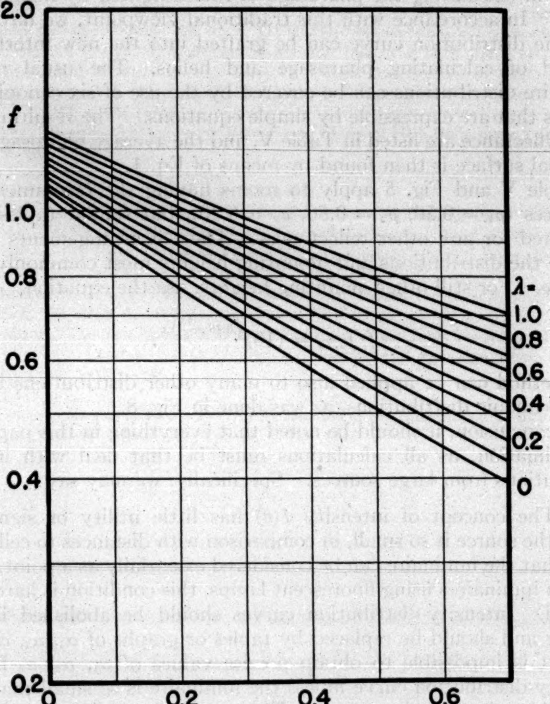

FIG. 8. Interflectance calculated for the eleven distributions of Fig. 7 with $h_m = \frac{3}{4}h$.

Of course, the better way is to discard intensity distributions entirely and to actually measure the pharos to floor, walls, and ceiling, as was done by Coradeschi and Innes (7). By employing experimental values of α_1, α_2, α_3 in Eq. 3, the designer obtains correct values of f, free from the ambiguity that is always associated with intensity distributions from large sources.

CONCLUSIONS

Intensity-distribution curves are still commonly employed in specifying the output of luminaires. It is traditional for engineers to use these curves in calculating the pharosage in rooms lighted by hanging luminaires. In accordance with this traditional viewpoint, we have shown how the distribution curve can be grafted into the new interflectance method of calculating pharosage and helios. The usual range of luminaire distributions can be covered by the use of six canonic distributions that are expressible by simple equations. The resulting values of interflectance are listed in Table V, and the average pharosage on the principal surface is then found by means of Eq. 1.

Table V and Fig. 5 apply to rooms having the recommended reflectances ($\rho_1 = 0.50$, $\rho_2 = 0.80$, $\rho_3 = 0.30$), but similar tables can be calculated for any other reflectances. The ten arrangements (Fig. 1) include the distributions and mounting heights most commonly met in practice. For still other mounting heights, use the equation,

$$f/f_{0.75} = (10)^{\left(0.75 - \frac{h_m}{h}\right)\zeta}. \tag{18}$$

The method can be applied also to many other distributions by combining canonic distributions, as was done in Fig. 8.

In conclusion, it should be noted that everything in this paper is an approximation, as all calculations must be that deal with intensity distributions from large sources. Specifically, we may say:

1. The concept of intensity $I(\theta)$ has little utility or significance unless the source is so small, in comparison with distances to ceiling and floor, that the luminaire can be considered essentially as a point. With modern luminaires using fluorescent lamps, this condition is hardly ever realized. Intensity-distribution curves should be abolished in room lighting and should be replaced by tables or graphs of α_1, α_2, α_3 *vs.* k_r.
2. It is impossible to obtain precise values of α_1, α_2, α_3 from an intensity-distribution curve unless the luminaire is so small that it can be considered as a point source. The precise determination of the α's necessitates *new experimental data*.
3. No lighting fixtures have ever been designed. Like Topsy, they "just growed," and their distribution curves are just what happened to result. In no case do they give uniformity in pharosage or in any other characteristic. Obviously, the luminaires of the future should produce a predetermined distribution with known α's. No intensity distribution curve is needed. An example of a luminaire, available today with predetermined α's, is the luminous ceiling (4).

4. We have indicated why the intensity distribution should be abolished and have shown that results obtained with it are necessarily approximate. If the engineer wishes to use it, however, he will find that the method of the present paper allows calculations to be made, usually within the accuracy limits of illumination practice.

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Insecticide Research.—The season for fighting insects is here and with it a control problem of increasing importance. Scientists have found that widespread use of DDT and other powerful insecticides may backfire on us.

DDT, for instance, kills off a majority of insect pests but it also selects hardy survivors as parents for next year's insect crop. By using DDT now, we are helping to breed "super bugs" for the future.

So, is it back to the fly swatter, or are insecticides here to stay?

Two University of Wisconsin zoologists, Dr. James F. Crow and Richard Weiner, have some answers to this question in recently completed research on insecticide resistance.

Their findings, published in *Science* magazine, show that resistance to DDT is not necessarily accompanied by resistance to other insecticides. Insects whose parents survived DDT are readily killed by certain other insecticides, if not by DDT.

Evidence for this is based on research with fruit flies. Although fruit flies are not common household or barnyard pests, they are insects commonly chosen for laboratory study. Much is known about the heredity of fruit flies. The backlog of information about fruit flies aided UW researchers in studying the inheritance of insecticide resistance.

Crow and Weiner approached their problem in this way:

Thousands of fruit flies, representing many generations, were treated with DDT. At first, the doses of DDT were small, but, as the flies became more resistant, the doses were increased. At the end of a year, offspring of treated flies withstood doses which killed an overwhelming majority of normal flies.

V.
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Then, the DDT treated flies were ready for tests with other insecticides. Ten insecticides were chosen. Five of them were related to DDT in that they contain the chemical, chlorine, just as DDT does. The other five did not contain chlorine.

"In comparing the number of flies killed by the two groups of insecticides, we found a significant difference," Dr. Crow says. "The insecticides with no chlorine killed both DDT treated flies and normal flies in almost equal numbers. On the other hand, insecticides like DDT did a poorer job of killing than they would normally."

The practical point of these studies is this:

By not using one kind of insecticide exclusively, we can do a better job of killing insect pests in the long run.

In this way, there is not as much danger of developing large numbers of insects resistant to one insecticide and its chemical relatives, the researchers point out.

Another question remains: Will insects eventually become resistant to all kinds of insecticides so that we truly will have to return to the fly swatter?

From a hereditary point of view, Dr. Crow doesn't think so. Resistant insects are not caused by insecticides. What insecticides do is select hardy insects already present as the parents of future insects.

Nor are the insects in nature apt to become as resistant as the fruit flies treated in the laboratory. Insects around the house or the barnyard, for instance, are not in constant contact with insecticides.

What may be a good insecticide now may not be good 10 years from now. It all depends on how constantly it is used. New insecticides probably will always be in demand, as no one is likely to be an all-time cure-all.

THE EFFECT OF GRADUAL LIGHT ABSORPTION IN A FILM OF NON-NEGLIGIBLE THICKNESS

GENERAL FORMULAS. THEIR APPLICATION TO AN EXPONENTIAL EMULSION UNDER THE ONE-QUANTIC AND THE TWO-QUANTIC HYPOTHESIS. GRAPHICAL CONSTRUCTION OF GAMMA AS MODIFIED BY GRADUAL ABSORPTION FOR ANY CHARACTERISTIC CURVE BELONGING TO A THIN EMULSION LAYER.

BY

LUDWIK SILBERSTEIN[1]

For thin emulsion coatings in which the exposure at the top face of the layer does not differ perceptibly from that of its bottom face, we can assume that all the grains present receive one and the same exposure E, say in ergs/cm.^{2}, or n quanta per unit area of the film. Not so, however, for coatings of considerable thickness, when grains receive an exposure decreasing with increasing depth of their location below the top surface.

Let δ be thickness of an optically uniform emulsion layer, x the depth or distance of a point reckoned from the top surface, E_0 the exposure at the top surface ($x = 0$), and E_x the exposure at any depth x. We shall assume the validity of the simple absorption or attenuation law, *viz.:*

$$E_x = E_0 e^{-\kappa x}, \tag{1}$$

where κ is a constant, whose numerical value will depend, of course, on the wave length of the incident radiation.

Let $dD = F(E_x)dx$ be the density produced by the exposure E_x in an infinitesimal layer of thickness dx placed at a depth x. Then the density produced in the whole emulsion layer by an exposure E_0, decreasing with depth according to (1), will be

$$D = \int_0^\delta F(E_x)dx, \tag{2}$$

where F is some function of E_x, of a form to be specified later on.

Instead of x, introduce as integration variable, $z = E_0 e^{-\kappa x}$. Then $dx = -\dfrac{dz}{\kappa z}$ and formula (2) will become

$$D = -\frac{1}{\kappa}\int_{E_0}^{E_\delta} F(z)\frac{dz}{z},$$

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[1] Deceased: formerly Scientific Consultant, Rochester, N. Y.

where $E_\delta = E_0 e^{-\kappa\delta}$. The transparency or transmission of the whole layer for radiation of a given wave length is, by definition,

$$T = \frac{E_\delta}{E_0}.$$

Thus, and if we henceforth write E instead of E_0, our general density formula will be

$$D = \frac{1}{\kappa} \int_{TE}^{E} F(z) \frac{dz}{z}, \tag{2'}$$

where $T < 1$ is a given number and $F(z)$ some function of its argument whose form need not, so far, be specified.

On the other hand, if the absorption is negligible, we may write, in formula (2), $E_z = E_0 \equiv E$. Thus, and denoting the corresponding density by D_0 (the subscript signifying "no absorption"), we have

$$D_0 = F(E)_\delta. \tag{2$_0$}$$

Thence, our general formula becomes

$$D = \frac{1}{\kappa_\delta} \int_{TE}^{E} D_0(z) \frac{dz}{z}.$$

But the transparency T is defined as E_δ/E or $e^{-\kappa\delta}$, and, therefore, $\kappa_\delta = -\log T$ or $\kappa_\delta = \log \frac{1}{T} = \log$ (opacity).

Such being the case, our last formula may be written

$$D = -\frac{1}{\log T} \int_{TE}^{E} D_0(z) \frac{dz}{z}, \tag{3'}$$

where T is the transparency of the whole emulsion layer, and thus some proper fraction, $T < 1$, and $D_0(z)$ the density produced in the whole layer by any exposure z or E, *ceteris paribus*, but in absence of absorption. Thence, if D_0 be given, analytically or graphically, for any exposure, the density D in a thick layer is found by the quadrature indicated by (3').

The characteristic curve of the absorptionless film is the plot of D_0 against $\log z = M \log z = x$, say. Now $dx = M\frac{dz}{z}$, $\frac{dz}{z} = \frac{dx}{M}$, and $M \log T = \log T$, the Briggsian logarithm of the transparency, so that the last formula may be conveniently written

$$D = -\frac{1}{\log T} \int_{(TE)}^{\log E} D_0(x) dx. \tag{3}$$

The meaning of this perfectly general formula can be stated as follows:

If the characteristic curve D_0, $x = \log$ exposure of the thin[2] film is given, say graphically, the ordinate $D = D(E)$ of the characteristic curve for the thick film (of the transparency T) is equal to *the area* below that curve, from TE to E, divided by $\log \frac{1}{T} = \log$ (opacity) or by the density of the unexposed thick film (see Fig. 1, where the area in question is shaded).

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[2] Absorptionless.

Notice that this result is wholly independent of our ability, or inability, to represent the function $D_0(x)$ analytically. If this function be given but graphically as is the case in every photographic research laboratory, the required density $D = D(E)$ is found, for any desired exposure E, by means of a planimeter which gives the value of the integral $\int_{\log(TE)}^{\log E} D_0 dx$. It remains only to divide it by $\log \frac{1}{T}$ or by the given density of the unexposed (or exposed, but not developed) film.

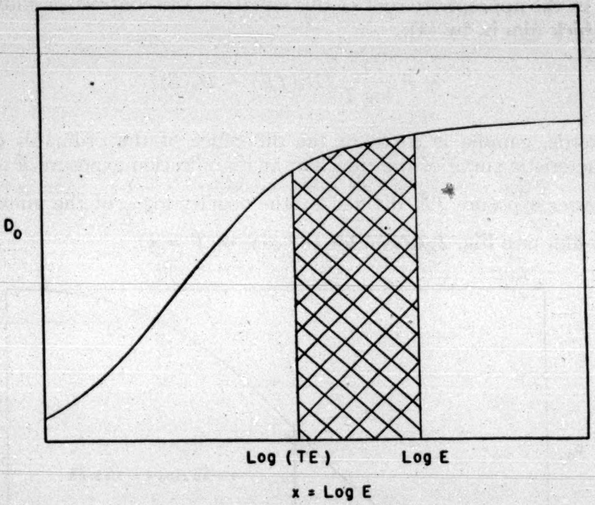

FIG. 1.

A simple implication of our formula (3) is the gradient $g = \frac{dD}{d \log E} = \frac{dD}{dx}$ of the characteristic curve of the thick film, *viz.*:

$$g = \frac{1}{\log T} \{ D_0(\log TE) - D_0(\log E) \}. \tag{4}$$

In words, the gradient of the curve in question, for the thick film at a point E, is equal to the difference of ordinates at the points E and TE of the curve belonging to the corresponding thin film, divided by the opacity, $\log \frac{1}{T}$, of the unexposed thick film. This, then, gives a very simple graphical construction of the gradient $g = g(E)$ from the drawn characteristic curve of the thin (absorptionless) film.

The gradient g acquires its maximum $g = g(\bar{E}) = \gamma$, where $dg/d \log E = 0$, that is to say, the inflection point of the D, $\log E$ curve is given by the equation

$$TD_0'(T\bar{E}) = D'(\bar{E}). \tag{5}$$

If $\bar{E}$ be the appropriate root of this equation, the contrast or gamma of the thick film is, by (4),

$$\gamma = \frac{1}{\log T}\{D_0(T\bar{E}) - D_0(\bar{E})\}. \tag{6}$$

In words, gamma is equal to the difference of the ordinates of the characteristic curve of the thin film at its inflection exposure $\bar{E}$ and at the lesser exposure $T\bar{E}$, divided by the opacity $\log \frac{1}{T}$ of the unexposed thick film (see Fig. 2, drawn for the case of $T = \frac{1}{2}$).

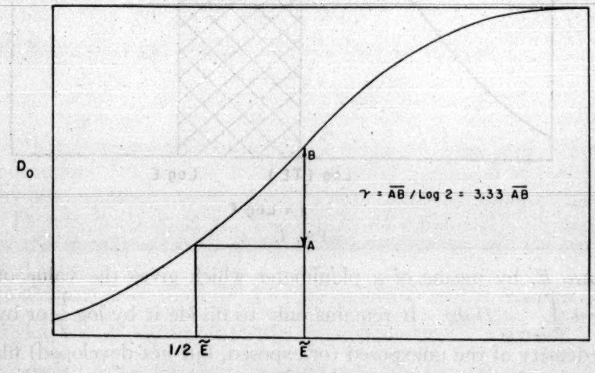

FIG. 2.

All these results hold good for any form whatever of the function $D_0 = D_0(x)$ appearing in the formula (3), that is, of the shape of the characteristic curve of a thin film corresponding to the actual thick film.

Let us now apply the general formula (3) to some particular forms of the function $D_0(x)$ appearing under the integral in that formula.

Uniform Emulsion: r = 1

Let a be the size (area) of each grain. Then

$$D_0(E) = D_m[1 - e^{-n\epsilon a}],\ n = \frac{\lambda}{hc}E$$

or

$$D_0(E) = D_m[1 - e^{-\alpha E}],$$

where

$$\alpha = \frac{\epsilon\lambda a}{hc}.$$

Thence, by (3),

$$D = \frac{D_m}{\log T}\int_{\log E}^{\log(TE)}(1 - e^{-\alpha E})dx,\ E = e^{x/M},$$

or

$$D = \frac{D_m}{\log T}\int_{E}^{TE}(1 - e^{-\alpha E})\frac{dE}{E}.$$

The first part of the integral is equal to $\log(TE) - \log E = \log T$. Thus

$$\frac{D}{D_m} = 1 - \frac{1}{\log T}\int_{E}^{TE} e^{-\alpha E}\frac{dE}{E}$$

or, substituting $u = \alpha E$,

$$\frac{D}{D_m} = 1 - \frac{1}{\log T}\int_{\alpha E}^{\alpha TE} e^{-u}\frac{du}{u}.$$

Finally, since

$$\alpha E = \frac{\lambda}{hc}\epsilon a E = na\epsilon = y,$$

in our notation, the required density-exposure formula is

$$D = D_m\left\{1 - \frac{1}{\log T}\int_{y}^{Ty}\frac{1}{u}e^{-u}\,du\right\}. \tag{7}$$

The integral is not reducible, in finite form, to any well-established functions. It is the difference of the so-called *exponential-integral*, which we may denote by E[*] for the arguments Ty and y, the limits of our integral. Thus, in terms of the exponential-integral, for which convenient numerical tables are available, we have

$$D = D_m\left\{1 + \frac{1}{\log T}[\mathsf{E}(y) - \mathsf{E}(Ty)]\right\}. \tag{7a}$$

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[*] It is usually denoted by E, which, however, could be confounded with the exposure, and it is better replaced by the Gothic E.

We may, also, develop the integral appearing in (7) into a power series, *viz.*:

$$\int_y^{Ty} \frac{1}{u}\left(1 - u + \frac{u^2}{2!} - \frac{u^3}{3!} + \dots\right) du$$

$$= \log T + (1 - T)y - \frac{1}{2\cdot 2!}(1 - T^2)y^2 + \frac{1}{3\cdot 3!}(1 - T^3)y^3 - \dots.$$

This gives

$$\frac{D}{D_m} = -\frac{1}{\log T}\left\{(1 - T)y - \frac{1}{2\cdot 2!}(1 - T^2)y^2 + \frac{1}{3\cdot 3!}(1 - T^3)y^3 - \dots\right\}. \qquad (7b)$$

Since $T < 1$, $\log T$ is negative, so that the constant factor of the infinite power series is positive, as it should be.

For $y = 0$, the formula (7*b*) gives $D = O$. For small exposures, that is, for $y \ll 1$, we have, up to y^2-terms,

$$D = \frac{D_m(1 - T)y}{\log O}, \qquad (7c)$$

where $O = 1/T$ is the "opacity" of the unexposed film. The density is thus initially proportional to the first power of the exposure, precisely as in the absorptionless film, for which we have $D_0 = D_m(1 - e^{-y})$ and, up to y^2- and higher terms,

$$D_0 = D_m y.$$

Only the coefficient of proportionality for the thick film is different. The ratio of the two densities is

$$\frac{D}{D_0} = \frac{1 - T}{-\log T} = \frac{T - 1}{\log T}.$$

If, for example, $T = \frac{1}{2}$, $-\log T = 0.301$, hence $-\log T = 0.692$, and

$$\frac{D}{D_0} = \frac{1}{13.84} \div \frac{1}{14}.$$

For $T = \frac{3}{4}$, we have $-\log T = 0.287$, so that

$$\frac{D}{D_0} = \frac{1}{1.15} \div 0.87,$$

and so on. For evanescent opacity, that is,

$$T = 1 - \tau, \qquad \text{where} \qquad \tau \ll 1,$$

we have $1 - T = \tau$, $\log T = \log(1 - \tau) = -\tau$, and, therefore,

$$\frac{D}{D_0} = \frac{\tau}{\tau} = 1.$$

To discover the effect of a low opacity, we must push the approximation one step further, including the terms in τ^2. Now $T - 1 = -\tau$, as before, but $\log T = -\tau - \frac{\tau^2}{2}$; hence

$$\frac{D}{D_0} = \frac{1}{1 + \frac{1}{2}\tau^2} \doteqdot 1 - \tfrac{1}{2}\tau^2, \tag{7d}$$

for example, for $T = 0.9$, $\tau = 0.1$, and $\frac{D}{D_0} = 0.995$, a slight depression of density due to the opacity, $O = \frac{1}{0.9} = 1.11$ of the (unexposed) film.

So far, we have seen the effect of gradual absorption in the case of weak exposures, small y. The series in (7*b*) converges rapidly enough for considerable exposures, such as $y = 1$, when

$$\frac{D(1)}{D_m} = -\frac{1 - T}{\log T}\left\{1 - \tfrac{1}{2}(1 + T) + \frac{1}{18}(1 + T + T^2) - \frac{1}{96}(1 + T)(1 + T^2) + \dots\right\}.$$

Thus, for example, for $T = \frac{1}{2}$, to three decimals,

$$\frac{D(1)}{D_m} = 0.327,$$

while, for the absorptionless film,

$$\frac{D_0(1)}{D_m} = 1 - \frac{1}{e} = 0.368,$$

and $D(1){:}D_0(1) = 0.894$. That is to say, a considerable opacity, $O = 2$, of the film reduces the density, at an exposure $y = 1$, almost by 11 per cent.

For yet greater exposures, such as $y = 2, 4$, etc., the formula (7*a*) must be used, to be aided by a table of the exponential-integral. For $y \to \infty$ the second, bracketed, term in (7*a*) tends to zero and D tends to D_m, as was to be expected.

The gradient of the characteristic curve of the thick film is given by (4), where, in our case,

$$D_0(E) = D_m(1 - e^{-y}),$$
$$D_0(TE) = D_m(1 - e^{-Ty}).$$

Thus

$$\frac{g}{D_m} = \frac{e^{-y}}{\log T}\{1 - e^{(1-T)y}\}. \tag{8}$$

For small y, the bracketed expression reduces to $-(1 - T)y$, so that, up to y^2-terms,

$$\frac{g}{D_m} = \frac{(1 - T)y}{\log O} = \frac{D}{D_m},$$

by (7*a*). The inflection point is, by (5), a root of the equation

$$Te^{-T\bar{y}} = e^{-\bar{y}} \quad \text{or} \quad e^{-(1-T)\bar{y}} = T,$$

whence

$$\bar{y} = \frac{\log O}{1 - T}, \quad O = \frac{1}{T}. \tag{9}$$

Thence, by (8),

$$\gamma/D_m = \frac{e^{-\bar{y}}}{\log T}\{1 - e^{-\log T}\}$$

and since, by (9),

$$e^{-\bar{y}} = T^{-M/(1-T)},$$

we have, ultimately,

$$\gamma/D_m = \frac{O^{M/(1-T)}}{\log T}\{1 - O^M\}. \tag{10}$$

If $T = 1 - \tau$, $\tau \ll 1$ (as before), then, after simple reductions,

$$\gamma/D_m = \frac{\tau^{M-1}}{M(1 - \tau)^{M/\tau}}, \tag{10a}$$

for example, for $T = 0.9$ or $\tau = 0.1$,

$$\gamma/D_m = \frac{10^{1-M}}{0.9^{10M}} = \frac{10}{9^{10M}} = \frac{10}{9^{4.343}}.$$

For an *almost transparent* film, $T = 1 - \tau(\tau \ll 1)$, formula (6) becomes, up to terms in τ^2,

$$\gamma = \frac{\tau\bar{E}D_0'(\bar{E})}{\log O} = \frac{\bar{E}}{M}D_0'(\bar{E}), \tag{a}$$

where $\bar{E}$ is to be determined from Eq. 5 which in our case reduces to

$$D_0'(\bar{E}) + \bar{E}D_0''(\bar{E}) = 0. \tag{b}$$

From (*a*) and (*b*) we see that the inflection point and the gradient at this point, or gamma, are *up to* τ^2*-terms* unaffected by absorption. In order to detect the effect of absorption on $\bar{E}$ and γ, we must retain the terms in τ^2. Now,

$$D_0(\bar{E} - \tau\bar{E}) - D_0(\bar{E}) = -\tau\bar{E}D_0'(\bar{E}) + \frac{\tau^2\bar{E}^2}{2}D_0''(\bar{E})$$

and

$$\log T = M \log (1 - \tau) = M\left(-\tau + \frac{\tau^2}{2}\right).$$

Thus the general formula (6) becomes

$$\gamma = \frac{\bar{E}}{M}\left\{D_0'(\bar{E}) - \frac{\tau}{2}[\bar{E}D_0''(\bar{E}) - D_0'(\bar{E})]\right\}, \tag{11}$$

and the Eq. 5 for $\bar{E}$,

$$D_0'(\bar{E}) + \bar{E}\, D_0''(\bar{E}) + \tau\bar{E}\left[D_0''(\bar{E}) + \frac{\bar{E}}{2} D_0'''(\bar{E})\right] = 0. \tag{12}$$

For $\tau = 0$ we replace in (*a*), (*b*) the γ, $\bar{E}$-values for the absorptionless film. The terms with τ in (11) and (12) give the effect (upon γ and the inflection point) of a slight absorption. The coefficients of τ in (11) and (12) are instantly determined, when the function form $D_0(E)$ or the characteristic curve of the absorptionless film is prescribed. The reader may verify that (12) with (11) leads, in the case of $D_0(E) = D_m[1 - e^{-\alpha E}]$, to $\bar{y} = \dfrac{\log T}{T - 1}$ and $\gamma/D_m = \tau^{M-1}/M(1 - \tau)^{M/\tau}$, as in (9) and (10*a*).

Equations 11 and 12 may be rewritten with y replacing E, since y/E is a constant. Thus we have, for a quasi-transparent film ($T = 1 - \tau$, $\tau \ll 1$), and for $D_0(y)$ a function of any form whatever, the equation for $\bar{y}$,

$$D_0'(y) + E\, D_0''(y) + \tau y\left[D_0''(y) + \frac{y}{2} D_0'''(y)\right] = 0, \tag{12a}$$

and the expression for γ,

$$\gamma = \frac{\bar{y}}{M}\left\{D_0'(y) - \frac{\tau}{2}[\bar{y}\, D_0''(\bar{y}) - D_0'(\bar{y})]\right\}. \tag{11a}$$

Let us now return to the perfectly general expression (3) for the density produced by an exposure E in any thick film whatever (notice that (3) is wholly independent of the quantic or any other theory of exposure) and let us apply it to an *exponential* emulsion, $f(a) = e^{-a}$, under the *one*-quantic assumption, that is to say,

$$D(y) = D_m \frac{y(2 + y)}{(1 + y)^2}.$$

Since $y = n\bar{a}\epsilon = \dfrac{\lambda\bar{a}\epsilon}{hc} E = \alpha E$, (say), we have $x = \log E = \log \dfrac{y}{\alpha}$, $dx = M\dfrac{dy}{y}$,

$$D = -\frac{MD_m}{\log T}\int_{Ty}^{y} \frac{2 + y}{(1 + y)^2}\, dy.$$

The indefinite integral (or primitive) of $\frac{2+y}{(1+y)^2}$ is

$$\log(1+y) - \frac{1}{1+y}.$$

Thence, the required density-exposure formula, for $r = 1$,

$$\frac{D}{D_m} = \frac{1}{\log T}\left\{\frac{1}{1+y} - \frac{1}{1+Ty} + \log\frac{1+Ty}{1+y}\right\}. \tag{13·1}$$

Under the two-quantic hypothesis ($r = 2$), for an exponential emulsion, as before, we have

$$\frac{D_0}{D_m} = \frac{y^2(3+y)}{(1+y)^3}$$

and

$$\log T\cdot\frac{D}{D_m} = \int_y^{Ty}\frac{y(3+y)}{(1+y)^3}\,dy.$$

The integral is readily evaluated, by partial integrations, and the result is

$$\log T\cdot\frac{D}{D_m} = \left|\log(1+y) - \frac{3+8y+3y^2}{2(1+y)^2}\right|_y^{Ty}$$

or

$$\frac{D}{D_m} = \frac{1}{\log T}\left\{\log\frac{1+Ty}{1+y} - \frac{3+8Ty+3T^2y^2}{2(1+Ty)^2} + \frac{3+8y+3y^2}{2(1+y)^2}\right\}. \tag{13·2}$$

In much the same way, the integral in the expression for the density,

$$\frac{D}{D_m} = \frac{1}{\log T}\int_y^{Ty}\frac{y^{r-1}(r+1+y)}{(1+y)^{r+1}}\,dy \tag{13·r}$$

in a thick coating of an exponential emulsion, for any value of the quantum number r can be evaluated, *viz.*, by a succession of partial integrations. The expression contains in each case a rational algebraic function of the exposure variable y and a logarithmic term $\log\frac{1+Ty}{1+y}$.

This exhausts, in essence, the question of the effect of the gradual absorption of light upon the density produced by any exposure of a thick emulsion coating.

V. 19

SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS (WITH TABLES)

BY

LOUIS WEINBERG[1]

INTRODUCTION

A large number of problems in mathematical physics can be described by the following partial differential equations:

(*a*) Laplace's equation, $\nabla^2\phi = 0$

(*b*) Poisson's equation, $\nabla^2\phi = -K(u^1, u^2, u^3)$, where K is a known function of the coordinates, or a constant

(*c*) Diffusion equation, $\nabla^2\phi = \dfrac{1}{h^2}\dfrac{\partial\phi}{\partial t}$

(*d*) Wave equation, $\nabla^2\phi = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$

(*e*) Damped-wave equation, $\nabla^2\phi = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} + R\dfrac{\partial\phi}{\partial t}$

(*f*) Helmholtz equation, $\nabla^2\phi + \gamma^2\phi = 0$, where γ^2 is a constant.

The conventional method for the solution of the partial differential equations of mathematical physics is by separation of variables. The substitution $\phi = U^1(u^1)U^2(u^2)U^3(u^3)$ is made, which, after some manipulation and the introduction of separation constants, allows the partial differential equation to be separated into three ordinary differential equations, each of which yields a U-solution. Since, with the exception of Poisson's equation, the equations are linear and homogeneous (and as will be shown later, Poisson's equation can be made homogeneous), any sum of solutions of the form $U^1U^2U^3$ is also a solution. The initial and boundary conditions of a particular problem are used in the evaluation of the constants so that a unique solution is obtained.

The usual method tends to be long and laborious, particularly in the more difficult coordinate systems where even the form of *div* and *curl* may not readily be available. The purpose of the present paper is to tabulate some necessary equations and solutions so the work will be expedited. The paper will be divided into three parts:

1. Each of the above equations will be shown to be a special case of the Helmholtz equation.

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2. The mechanics of finding solutions of the Helmholtz equation will be demonstrated.

3. Tables of solutions for eight of the most commonly used coordinate systems will be given.

THE HELMHOLTZ EQUATION

Consider the equations listed in the first paragraph:

(*a*) It is evident that Laplace's equation,

$$\nabla^2\phi = 0, \tag{1}$$

is a special case of the Helmholtz equation,

$$\nabla^2\phi + \gamma^2\phi = 0, \tag{2}$$

where $\gamma^2 = 0$. Therefore a solution of the Helmholtz equation for the limit $\gamma \to 0$ gives a solution of Laplace's equation.

(*b*) Poisson's equation can be reduced to Laplace's equation by a change of dependent variable. Let

$$\phi = \Omega + g_1(u^1, u^2, u^3), \tag{3}$$

with g_1 so chosen that $\nabla^2\Omega = 0$.

A simple case of $\phi = \phi(u^1)$ is used to illustrate the method. Suppose

$$\nabla^2\phi = \frac{\rho}{\epsilon r} \tag{4}$$

for a circular-cylinder system, where $\phi = \phi(r)$ and ρ is a constant. From Table II,[2] the Laplacian is

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}. \tag{5}$$

Let

$$\phi = \Omega + Ar; \tag{6}$$

then

$$\left.\begin{aligned} \frac{1}{r}\frac{d\phi}{dr} &= \frac{1}{r}\frac{d\Omega}{dr} + \frac{A}{r} \\ \frac{d^2\phi}{dr^2} &= \frac{d^2\Omega}{dr^2} \end{aligned}\right\}. \tag{7}$$

Substitution of Eqs. 5 and 7 in Eq. 4 gives

$$\frac{d^2\Omega}{dr^2} + \frac{1}{r}\frac{d\Omega}{dr} + \frac{A}{r} - \frac{\rho}{\epsilon r} = 0. \tag{8}$$

It is therefore clear that the substitution

$$\phi = \Omega + \frac{\rho r}{\epsilon} \tag{9}$$

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[2] See Appendix for all tables.

reduces Eq. 4 to Laplace's equation

$$\nabla^2\Omega = 0, \tag{10}$$

whose solution gives ϕ by use of Eq. 9.

For a more complicated example, consider

$$\nabla^2\phi = -\rho \tag{11}$$

in elliptic-cylinder coordinates, where ρ is a constant and ϕ is a function of μ and ν only. From Table III (η of the table corresponds to μ and θ to ν), the Laplacian is obtained so that Poisson's equation becomes

$$\frac{1}{c^2(\cosh^2\mu - \cos^2\nu)}\left[\frac{\partial^2\phi}{\partial\mu^2} + \frac{\partial^2\phi}{\partial\nu^2}\right] = -\rho. \tag{12}$$

Cross multiplication yields

$$\begin{aligned}\frac{\partial^2\phi}{\partial\mu^2} + \frac{\partial^2\phi}{\partial\nu^2} &= -\rho c^2[\cosh^2\mu - \cos^2\nu] \\ &= -\frac{\rho c^2}{2}[\cosh 2\mu - \cos 2\nu].\end{aligned} \tag{13}$$

Let

$$\phi = \Omega + g_1(\mu) + g_2(\nu); \tag{14}$$

then

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial\mu^2} &= \frac{\partial^2\Omega}{\partial\mu^2} + \frac{d^2g_1}{d\mu^2} \\ \frac{\partial^2\phi}{\partial\nu^2} &= \frac{\partial^2\Omega}{\partial\nu^2} + \frac{d^2g_2}{d\nu^2}\end{aligned}\right\}. \tag{15}$$

Substitution of Eq. 15 in Eq. 13 gives

$$\begin{aligned}\frac{\partial^2\Omega}{\partial\mu^2} + \frac{\partial^2\Omega}{\partial\nu^2} &+ \left[\frac{d^2g_1}{d\mu^2} + \frac{\rho c^2}{2}\cosh 2\mu\right] \\ &+ \left[\frac{d^2g_2}{d\nu^2} - \frac{\rho c^2}{2}\cos 2\nu\right] = 0.\end{aligned} \tag{16}$$

Laplace's equation results if each of the two brackets is set equal to zero, or

$$\left.\begin{aligned}\frac{d^2g_1}{d\mu^2} &= -\frac{\rho c^2}{2}\cosh 2\mu \\ \frac{d^2g_2}{d\nu^2} &= \frac{\rho c^2}{2}\cos 2\nu\end{aligned}\right\}. \tag{17}$$

Integration twice yields

$$\left.\begin{aligned}g_1(\mu) &= -\frac{\rho c^2}{8}\cosh 2\mu \\ g_2(\nu) &= -\frac{\rho c^2}{8}\cos 2\nu\end{aligned}\right\}, \tag{18}$$

so that

$$\phi = \Omega - \frac{\rho c^2}{8}(\cosh 2\mu + \cos 2\nu). \tag{19}$$

The solution of the problem is thus obtained by consulting Table III to find the solution to

$$\frac{\partial^2 \Omega}{\partial \mu^2} + \frac{\partial^2 \Omega}{\partial \nu^2} = 0, \tag{20}$$

which is

$$\Omega = e^{\pm m\mu} e^{\pm im\nu}$$

or alternate forms; and ϕ is then expressed by Eq. 19.

It is thus evident that solutions to the Helmholtz equation can be modified to yield solutions of Poisson's equation.

(*c*) The diffusion equation

$$\nabla^2 \phi = \frac{1}{h^2}\frac{\partial \phi}{\partial t} \tag{21}$$

gives a solution

$$\phi = U^1 U^2 U^3 T(t) = U(u^1, u^2, u^3) T(t), \tag{22}$$

where

$$T = e^{-h^2 \gamma^2 t}. \tag{23}$$

This is made evident by substitution of

$$\phi = UT \tag{24}$$

in Eq. 21 and division by UT, which result in

$$\frac{1}{U}\nabla^2 U = \frac{1}{h^2 T}\frac{dT}{dt}. \tag{25}$$

Since the first member is independent of t, and the second member is independent of the space coordinates, each can be set equal to a constant, $-\gamma^2$, yielding

$$\frac{1}{U}\nabla^2 U = \frac{1}{h^2 T}\frac{dT}{dt} = -\gamma^2. \tag{26}$$

The two separated equations are

$$\left.\begin{aligned} \frac{dT}{dt} + \gamma^2 h^2 T &= 0 \\ \nabla^2 U + \gamma^2 U &= 0 \end{aligned}\right\}, \tag{27}$$

the solution of the first of which is given in Eq. 23; the second is the Helmholtz equation.

Therefore a solution of the Helmholtz equation can be converted to a solution of the diffusion equation by multiplication by $e^{-h^2\gamma^2 t}$.

(*d*) A similar procedure is used to reduce the other two equations to the Helmholtz equation. For example, the wave equation, upon substitution of

$$\phi = UT, \tag{28}$$

becomes

$$\frac{1}{U}\nabla^2 U = \frac{1}{c^2 T}\frac{d^2 T}{dt^2} = -\gamma^2. \tag{29}$$

The separated time equation is always

$$\frac{1}{c^2}\frac{d^2 T}{dt^2} + \gamma^2 T = 0, \tag{30}$$

which gives as its contribution to the product solution

$$T = \frac{\sin}{\cos}(\gamma c t). \tag{31}$$

The other separated equation is again the Helmholtz equation.

(*e*) When the above procedure is used on the damped-wave equation the separated time equation is

$$\frac{d^2 T}{dt^2} + Rc^2\frac{dT}{dt} + \gamma^2 c^2 T = 0. \tag{32}$$

The solution of the above gives

$$T = e^{-\alpha \pm j\omega}, \tag{33}$$

where

$$\left.\begin{aligned} \alpha &= \frac{Rc^2}{2} \\ \text{and}\qquad \omega &= \sqrt{\gamma^2 c^2 - \frac{R^2 c^4}{4}} \end{aligned}\right\}. \tag{34}$$

From a consideration of the above analyses, the important conclusion is reached: *solutions to the Helmholtz equation provide solutions to all the equations listed at the beginning of the paper.*

SOLUTION OF HELMHOLTZ EQUATION

It is best to begin with a statement of the general separation equation. It is

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0, \tag{35}$$

where f_i are given functions of a single coordinate; U^i are the components of the product solution $\phi = U^1(u^1)U^2(u^2)U^3(u^3)$, which are to be found; u^i are the coordinate variables; α_j are the separation constants, which are determined by application of the boundary conditions; and

Φ_{ij} are the elements of the Stäckel determinant,[3]

$$S = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix} . \tag{36}$$

When the separation equation is used for solving the Helmholtz equation, the first separation constant α_1 is set equal to γ^2. In passing, it may be noted that $\alpha_1 = 0$ gives the separation equations for Laplace's equation directly.

The elements of the Stäckel determinant and the f_i are substituted in the general separation equation to give the three separated equations, which are then converted into forms whose solutions can be recognized. The method is best illustrated by an example.

ILLUSTRATIVE EXAMPLE

It is desired to solve the Helmholtz equation for a circular-cylinder coordinate system in which $\phi = \phi(r, \psi, z)$. In Table II it is found that

$$f_1 = r, \quad f_2 = f_3 = 1 \tag{37}$$

and

$$S = \begin{vmatrix} 0 & -\dfrac{1}{r^2} & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} . \tag{38}$$

Substitution from Eqs. 37 and 38 in the general separation equation, Eq. 35, yields the three separation equations:

$$\left.\begin{aligned} &\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + R\left(-\frac{1}{r^2}\alpha_2 - \alpha_3\right) = 0, \\ &\frac{d^2\Psi}{d\psi^2} + \alpha_2\Psi = 0, \\ &\frac{d^2Z}{dz^2} + (\gamma^2 + \alpha_3)Z = 0, \end{aligned}\right\} \tag{39}$$

where $U^1 = R$, $U^2 = \Psi$, and $U^3 = Z$.

For convenience, let

$$\left.\begin{aligned} \alpha_2 &= m^2 \\ \alpha_3 &= -\beta^2 \end{aligned}\right\}. \tag{40}$$

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[3] To carry out the mechanics of obtaining the solution, one need not know how to determine the f_i and Stäckel determinant. They will be given for each coordinate system. However, for those who wish to know, see PARRY MOON, "Notes on Field Theory," Course 6.586, Massachusetts Institute of Technology (unpublished).

The solutions of the separated equations are recognized and give as the product solution

$$\phi = R\Psi Z = \frac{J_m}{N_m}(\beta r) \frac{\sin}{\cos}(m\psi) \frac{\sin}{\cos}(\sqrt{\gamma^2 - \beta^2}z), \tag{41}$$

where J_m and N_m are Bessel functions of the first and second kinds, respectively.

If the original problem deals with the wave equation, the solution is

$$\phi = R\Psi Z \frac{\sin}{\cos}(\gamma ct), \tag{42}$$

while for the diffusion equation

$$\phi = R\Psi Z e^{-k^2\gamma^2 t}. \tag{43}$$

USE OF THE TABLES

The student laboriously learns to integrate; but later in life, he uses a table of integrals to expedite his work. Similarly, one can reduce the labor in solving field problems by employing the tables of solutions for the Helmholtz equation and Laplace's equation. Solutions are listed in the Appendix for the following coordinate systems: (1) rectangular, (2) circular cylinder, (3) elliptic cylinder, (4) parabolic cylinder, (5) spherical, (6) prolate spheroidal, (7) oblate spheroidal, and (8) parabolic.

For convenience, and to extend the sphere of usefulness of these tables, it was deemed advisable to list the solutions of both Laplace's equation and the Helmholtz equation for the cases of ϕ a function of only one or two coordinates under special solutions, and of all three coordinates as solutions for the general case. Additional data are given for each coordinate system, so that necessary information is assembled in one place.

SUMMARY

A simple method has been demonstrated for obtaining some solutions of the principal equations of mathematical physics. Tables of solutions have been included in the Appendix so that problems in field theory can be solved with a minimum of labor.

Acknowledgment

The author wishes to express his gratitude to Professor Parry Moon of the Massachusetts Institute of Technology for his great help and encouragement in the preparation of this paper. The convenient notation for the solutions arising in the spheroidal and parabolic coordinate systems is due to Professor Moon, and the brief discussion of $J_p(\gamma, qx)$ is taken from his unpublished notes.

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(3) L. J. CHU AND J. A. STRATTON, "Elliptic and Spheroidal Wave Functions," *J. Math. Phys.* Vol. 20, 1941, pp. 259, ff.

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APPENDIX

Introduction to the Tables

For each coordinate system there is a functional relationship between its coordinates and those of a rectangular system.

$$\begin{aligned} x^1 &= x^1(u^1, u^2, u^3), \\ x^2 &= x^2(u^1, u^2, u^3), \\ x^3 &= x^3(u^1, u^2, u^3). \end{aligned}$$

With the above information the metric coefficients g_{ii} can be obtained. They are important in the systematic transformation of coordinates. For orthogonal coordinates,

$$g_{ii} = \left(\frac{\partial x^1}{\partial u^i}\right)^2 + \left(\frac{\partial x^2}{\partial u^i}\right)^2 + \left(\frac{\partial x^3}{\partial u^i}\right)^2, \qquad i = 1, 2, 3.$$

The element of length in a coordinate direction is $\sqrt{g_{ii}}\, du^i$; the element of area is $\sqrt{g_{11}g_{22}}\, du^1\, du^2$, or $\sqrt{g_{11}g_{33}}\, du^1\, du^3$, etc.; the element of volume is

$$\begin{aligned} d\tau &= \sqrt{g_{11}g_{22}g_{33}}\, du^1\, du^2\, du^3 \\ &= \sqrt{g}\, du^1\, du^2\, du^3. \end{aligned}$$

The gradient of a scalar function ϕ is

$$\nabla\phi = \frac{\mathbf{a}_1}{\sqrt{g_{11}}}\frac{\partial\phi}{\partial u^1} + \frac{\mathbf{a}_2}{\sqrt{g_{22}}}\frac{\partial\phi}{\partial u^2} + \frac{\mathbf{a}_3}{\sqrt{g_{33}}}\frac{\partial\phi}{\partial u^3},$$

where $\mathbf{a}_i$ are unit vectors in a coordinate direction.

The Laplacian ∇^2 of a scalar function ϕ is

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\left\{\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\phi}{\partial u^i}\right)\right\}.$$

The divergence of a vector $\mathbf{E}$ is

$$\operatorname{div}\mathbf{E} = \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u^1}\left(\sqrt{\frac{g}{g_{11}}}\,E_1\right) + \frac{\partial}{\partial u^2}\left(\sqrt{\frac{g}{g_{22}}}\,E_2\right) + \frac{\partial}{\partial u^3}\left(\sqrt{\frac{g}{g_{33}}}\,E_3\right)\right\}.$$

The curl of a vector $\mathbf{E}$ is

$$\begin{aligned} \operatorname{curl}\mathbf{E} = \frac{1}{\sqrt{g}}\Big\{ &\mathbf{a}_1\sqrt{g_{11}}\left[\frac{\partial}{\partial u^2}(\sqrt{g_{33}}\,E_3) - \frac{\partial}{\partial u^3}(\sqrt{g_{22}}\,E_2)\right] \\ &+ \mathbf{a}_2\sqrt{g_{22}}\left[\frac{\partial}{\partial u^3}(\sqrt{g_{11}}\,E_1) - \frac{\partial}{\partial u^1}(\sqrt{g_{33}}\,E_3)\right] \\ &+ \mathbf{a}_3\sqrt{g_{33}}\left[\frac{\partial}{\partial u^1}(\sqrt{g_{22}}\,E_2) - \frac{\partial}{\partial u^2}(\sqrt{g_{11}}\,E_1)\right]\Big\}. \end{aligned}$$

Notation

Most of the functions obtained as solutions are familiar to engineers, and consequently no references have been given for them. Two classes of functions, however, *viz.*, $J_p(\gamma, x)$, $N_p(\gamma, x)$, and $P_p^q(\gamma, x)$, $Q_p^q(\gamma, x)$, are unfamiliar and difficult to use. The author has adopted what he considers a convenient notation for these functions; in learning to apply them to specific problems, one should consult the references cited where their characteristics are discussed in detail. As an illustration of such characteristics the series expansion and limiting forms for $J_p(\gamma, qx)$ and $J_p(\gamma, iqx)$ are discussed in this section.

i = $\sqrt{-1}$.
$\sim$ = asymptotically equal.
α_2, α_3 = separation constants.
γ^2 = constant appearing in Helmholtz equation.
p, q = any real numbers.
m, n = any real integers.
$J_p(x)$, $N_p(x)$ = Bessel functions of the first and second kinds, respectively, of order p and with argument x.
$P_p(x)$, $Q_p(x)$ = Legendre functions of the first and second kinds, respectively, of order p and with argument x.
$P_p^q(x)$, $Q_p^q(x)$ = associated Legendre functions of the first and second kinds, respectively.
$D_p(x)$, $D_{-p-1}(x)$ = Weber functions with argument x (see Ref. 1, p. 347).

$$\left.\begin{array}{l} Ce_m(x, q),\ Fe_m(x, q) \\ Se_m(x, q),\ Ge_m(x, q) \\ ce_m(x, q),\ fe_m(x, q) \\ se_m(x, q),\ ge_m(x, q) \end{array}\right\} = \text{Mathieu functions (Ref. 2; also Ref. 1, pp. 404, ff.).}$$

$P_p^q(\gamma, x)$, $Q_p^q(\gamma, x)$ = convenient notation for functions of the first and second kinds, respectively, with variable argument x and parameter γ. [These functions appear as solutions in spheroidal coordinates; they are discussed at length in Ref. 3, where they are designated by letters S and R.]
$J_p(\gamma, x)$, $N_p(\gamma, x)$ = convenient notation for functions of the first and second kinds, respectively, with variable argument x and parameter γ. [These functions appear as solutions in parabolic coordinates; the solutions are also given by other authors in terms of the confluent hypergeometric function, ${}_1F_1$, and the Whittaker function; also see discussion below.]
${}_1F_1$ = confluent hypergeometric function [Ref. 4, pp. 275, ff., where it is designated by M; Ref. 1, pp. 337–354].
$W_{k,m}(z)$ = Whittaker's function [Ref. 1, pp. 343, ff.].

Discussion of $J_p(\gamma, qx)$ and $J_p(\gamma, iqx)$, Solution of the Helmholtz Equation for Parabolic Coordinates

1. $J_p(\gamma, qx) = x^p \sum_{n=0}^{\infty} A_n x^{2n}$

where

$$A_0 = \frac{1}{\Gamma(p+1)},$$

$$A_1 = -\frac{1}{2!\Gamma(p+2)}\left(\frac{q}{2}\right)^2,$$

$$A_n = -\frac{(\gamma^2 A_{n-2} + q^2 A_{n-1})}{4n(p+n)}$$

and Γ is the Gamma function.

2. Special values of $J_p(\gamma, qx)$

$$\text{If } x = 0\ (p \neq 0),\ J_p(\gamma, qx) = 0.$$

$$\text{If } \gamma^2x^2 \gg \left| q^2 - \left(\frac{p}{x}\right)^2 \right|,\ J_p(\gamma, qx) \approx J_0\left(\frac{\gamma x^2}{2}\right).$$

$$\text{If } x \to \infty,\ J_p(\gamma, qx) \sim \frac{2}{x\sqrt{\pi\gamma}} \cos\left(\frac{\gamma x^2}{2} - \frac{\pi}{4}\right).$$

$$J_p(\gamma, iqx) = x^p \sum_{n=0}^{\infty} A_n x^{2n}$$

where

$$A_0 = \frac{1}{\Gamma(p+1)},$$

$$A_1 = \frac{1}{2!\Gamma(p+2)}\left(\frac{q}{2}\right)^2,$$

$$A_n = \frac{q^2A_{n-1} - \gamma^2A_{n-2}}{4n(p+n)}.$$

3. Special values of $J_p(\gamma, iqx)$

$$\text{If } x = 0\ (p \neq 0),\ J_p(\gamma, iqx) = 0.$$

$$\text{If } \gamma^2x^2 \gg \left| q^2 + \left(\frac{p}{x}\right)^2 \right|,\ J_p(\gamma, iqx) \approx J_0\left(\frac{\gamma x^2}{2}\right).$$

$$\text{If } x \to \infty,\ J_p(\gamma, iqx) \sim \frac{2}{x\sqrt{\pi\gamma}} \cos\left(\frac{\gamma x^2}{2} - \frac{\pi}{4}\right).$$

$$J_0(\gamma, qx) = \sum_{n=0}^{\infty} C_n x^{2n}$$

where

$$C_0 = 1,\ C_1 = -\left(\frac{q}{2}\right)^2,$$

$$C_n = -\frac{1}{(2n)^2}\left[\gamma^2C_{n-2} + q^2C_{n-1}\right]$$

or

$$J_0(\gamma, qx) = J_0(qx) + \frac{\gamma^2x^4}{16}\left[-1 + \frac{\gamma^2x^4}{64} + \cdots + \frac{1}{72}\left(\frac{qx}{2}\right)^2\left(5 - 7\left(\frac{qx}{2}\right)^2 + \cdots\right)\right].$$

4. Special values of $J_0(\gamma, qx)$

$$\text{If } x = 0,\ J_0(\gamma, qx) = J_0(qx) = 1.$$

$$\text{If } \gamma^2x^2 \gg q^2,\ J_0(\gamma, qx) \approx J_0\left(\frac{\gamma x^2}{2}\right).$$

$$\text{If } x \to \infty,\ J_0(\gamma, qx) \sim \frac{2}{x\sqrt{\pi\gamma}} \cos\left(\frac{\gamma x^2}{2} - \frac{\pi}{4}\right).$$

$$J_0(\gamma, iqx) = \sum_{n=0}^{\infty} C_n x^{2n}$$

where

$$C_0 = 1,\ C_1 = \left(\frac{q}{2}\right)^2,$$

$$C_n = \frac{1}{(2n)^2}\left[q^2 C_{n-1} - \gamma^2 C_{n-2}\right]$$

or

$$J_0(\gamma, iqx) = J_0(iqx) + \frac{\gamma^2 x^4}{16}\left[-1 + \frac{\gamma^2 x^4}{64} + \cdots - \frac{1}{72}\left(\frac{qx}{2}\right)^2\left(5 + 7\left(\frac{qx}{2}\right)^2 + \cdots\right)\right].$$

5. *Special values of* $J_0(\gamma, iqx)$

$$\text{If } x = 0,\ J_0(\gamma, iqx) = J_0(iqx) = 1.$$

$$\text{If } \gamma^2 x^2 \gg q^2,\ J_0(\gamma, iqx) \approx J_0(\gamma, qx) \approx J_0\left(\frac{\gamma x^2}{2}\right).$$

$$\text{If } x \to \infty,\ J_0(\gamma, iqx) \sim \frac{2}{x\sqrt{\pi\gamma}} \cos\left(\frac{\gamma x^2}{2} - \frac{\pi}{4}\right).$$

TABLE I.—*Rectangular Coordinates.*

$$u^1 = x,\ u^2 = y,\ u^3 = z$$

$$S = \begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$f_1 = f_2 = f_3 = 1$$

$$g_{11} = g_{22} = g_{33} = 1,\ \sqrt{g} = 1$$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$\nabla\phi = \mathbf{a}_x \frac{\partial\phi}{\partial x} + \mathbf{a}_y \frac{\partial\phi}{\partial y} + \mathbf{a}_z \frac{\partial\phi}{\partial z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\operatorname{curl} \mathbf{E} = \mathbf{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right).$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $(\alpha_2 = p^2,\ \alpha_3 = q^2)$

$$\frac{d^2X}{dx^2} - (p^2 + q^2)X = 0 \qquad X = Ae^{\sqrt{p^2+q^2}\,x} + Be^{-\sqrt{p^2+q^2}\,x}$$

$$\frac{d^2Y}{dy^2} + p^2Y = 0 \qquad Y = A \sin py + B \cos py$$

$$\frac{d^2Z}{dz^2} + q^2Z = 0 \qquad Z = A \sin qy + B \cos qy.$$

Special Solutions

$$X = Ae^{px} + Be^{-px}, \qquad Y = A \sin py + B \cos py$$

$$X = A \sin px + B \cos px,\ Y = Ae^{py} + Be^{-py}$$

$$X = A + Bx,\ Y = A + By,\ Z = A + Bz.$$

SOLUTIONS OF THE HELMHOLTZ EQUATION $(\alpha_2 = p^2, \alpha_3 = q^2)$

General Case

$$\frac{d^2X}{dx^2} - (p^2 + q^2)X = 0 \qquad X = Ae^{\sqrt{p^2+q^2}\,x} + Be^{-\sqrt{p^2+q^2}\,x}$$

$$\frac{d^2Y}{dy^2} + p^2Y = 0 \qquad Y = A \sin py + B \cos py$$

$$\frac{d^2Z}{dz^2} + (\gamma^2 + q^2)Z = 0 \qquad Z = A \sin (\sqrt{\gamma^2 + q^2}\, z) + B \cos (\sqrt{\gamma^2 + q^2}\, z).$$

Special Solutions

$X = A + Bx,\ Y = A + By,\ Z = A \sin \gamma z + B \cos \gamma z.$

$X = Ae^{px} + Be^{-px},\ Y = A \sin (\sqrt{\gamma^2 + p^2}\, y) + B \cos (\sqrt{\gamma^2 + p^2}\, y).$

$X = A \sin px + B \cos px,\ Y = A \sin (\sqrt{\gamma^2 - p^2}\, y) + B \cos (\sqrt{\gamma^2 - p^2}\, y).$

TABLE II.—*Circular-Cylinder Coordinates.*

$u^1 = r,\ u^2 = \psi,\ u^3 = z$

$$S = \begin{vmatrix} 0 & -\frac{1}{r^2} & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$x^1 = r \cos \psi$

$x^2 = r \sin \psi$

$x^3 = z$

$f_1 = r,\ f_2 = f_3 = 1$

$g_{11} = 1,\ g_{22} = r^2,\ g_{33} = 1,\ \sqrt{g} = r$

$(ds)^2 = (dr)^2 + r^2(d\psi)^2 + (dz)^2$

$$\nabla\phi = \mathbf{a}_r \frac{\partial\phi}{\partial r} + \frac{\mathbf{a}_\psi}{r} \frac{\partial\phi}{\partial\psi} + \mathbf{a}_z \frac{\partial\phi}{\partial z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{div}\,\mathbf{E} = \frac{E_r}{r} + \frac{\partial E_r}{\partial r} + \frac{1}{r}\frac{\partial E_\psi}{\partial\psi} + \frac{\partial E_z}{\partial z}$$

$$\text{curl}\,\mathbf{E} = \mathbf{a}_r \left(\frac{1}{r}\frac{\partial E_z}{\partial\psi} - \frac{\partial E_\psi}{\partial z}\right) + \mathbf{a}_\psi \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) + \mathbf{a}_z \left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\psi) - \frac{1}{r}\frac{\partial E_r}{\partial\psi}\right].$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $(\alpha^2 = p^2, \alpha_3 = -q^2)$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{p^2}{r^2} - q^2\right)R = 0 \qquad R = AJ_p(qr) + BN_p(qr)$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0 \qquad \Psi = A \sin p\psi + B \cos p\psi$$

$$\frac{d^2Z}{dz^2} - q^2Z = 0 \qquad Z = Ae^{qz} + Be^{-qz}.$$

Special Solutions

$R = AJ_0(iqr) + BN_0(iqr),\ \Psi = A + B\psi,\ Z = A \sin qz + B \cos qz.$

$R = A + B \ln r,\ \Psi = A + B\psi,\ Z = A + Bz.$

$R = Ar^p + Br^{-p},\ \Psi = A \sin p\psi + B \cos p\psi.$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case ($\alpha_2 = p^2,\ \alpha_3 = -q^2$)

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{p^2}{r^2} - q^2\right)R = 0 \qquad R = AJ_p(qr) + BN_p(qr).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0 \qquad \Psi = A\sin p\psi + B\cos p\psi.$$

$$\frac{d^2Z}{dz^2} + (\gamma^2 - q^2)Z = 0 \qquad Z = A\sin(\sqrt{\gamma^2 - q^2}\,z) + B\cos(\sqrt{\gamma^2 - q^2}\,z).$$

Special Solutions

$R = AJ_0(iqr) + BN_0(iqr),\ \Psi = A + B\psi,\ Z = A\sin(\sqrt{\gamma^2 + q^2}\,z) + \cos(\sqrt{\gamma^2 + q^2}\,z).$

$R = A + B\ln r,\ \Psi = A + B\psi,\ Z = A\sin\gamma z + B\cos\gamma z.$

$R = AJ_p(\gamma r) + BN_p(\gamma r),\ \Psi = A\sin p\psi + B\cos p\psi.$

$R = AJ_0(\gamma r) + BN_0(\gamma r),\ \Psi = A + B\psi.$

$R = AJ_0(qr) + BN_0(qr),\ Z = A\sin(\sqrt{\gamma^2 - q^2}\,z) + B\cos(\sqrt{\gamma^2 - q^2}\,z).$

TABLE III.—*Elliptic-Cylinder Coordinates.*

$$S = \begin{vmatrix} 0 & -1 & -c^2\cosh^2\eta \\ 0 & 1 & c^2\cos^2\theta \\ 1 & 0 & 1 \end{vmatrix} \qquad \begin{aligned} u^1 &= \eta,\ u^2 = \theta,\ u^3 = z \\ x^1 &= c\cosh\eta\cos\theta \\ x^2 &= c\sinh\eta\cos\theta \\ x^3 &= z \end{aligned}$$

$$f_1 = f_2 = f_3 = 1$$

$$g_{11} = g_{22} = c^2(\cosh^2\eta - \cos^2\theta)$$

$$g_{33} = 1,\ \sqrt{g} = c^2(\cosh^2\eta - \cos^2\theta)$$

$$(ds)^2 = c^2(\cosh^2\eta - \cos^2\theta)[(d\eta)^2 + (d\theta)^2] + (dz)^2$$

$$\nabla\phi = \frac{1}{c\sqrt{\cosh^2\eta - \cos^2\theta}}\left[\mathbf{a}_\eta\frac{\partial\phi}{\partial\eta} + \mathbf{a}_\theta\frac{\partial\phi}{\partial\theta}\right] + \mathbf{a}_z\frac{\partial\phi}{\partial z}$$

$$\nabla^2\phi = \frac{1}{c^2(\cosh^2\eta - \cos^2\theta)}\left(\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\theta^2}\right) + \frac{\partial^2\phi}{\partial z^2}$$

$$\operatorname{div}\mathbf{E} = \frac{1}{c(\cosh^2\eta - \cos^2\theta)}\left[\frac{\partial}{\partial\eta}(\sqrt{\cosh^2\eta - \cos^2\theta}\,E_\eta) + \frac{\partial}{\partial\theta}(\sqrt{\cosh^2\eta - \cos^2\theta}\,E_\theta)\right] + \frac{\partial E_z}{\partial z}$$

$$\begin{aligned} \operatorname{curl}\mathbf{E} = {} & \mathbf{a}_\eta\left[\frac{1}{c\sqrt{\cosh^2\eta - \cos^2\theta}}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z}\right] \\ & + \mathbf{a}_\theta\left[\frac{\partial E_\eta}{\partial z} - \frac{1}{c\sqrt{\cosh^2\eta - \cos^2\theta}}\frac{\partial E_z}{\partial\eta}\right] \\ & + \frac{\mathbf{a}_z}{c(\cosh^2\eta - \cos^2\theta)}\left[\frac{\partial}{\partial\eta}(\sqrt{\cosh^2\eta - \cos^2\theta}\,E_\theta) - \frac{\partial}{\partial\theta}(\sqrt{\cosh^2\eta - \cos^2\theta}\,E_\eta)\right]. \end{aligned}$$

SOLUTIONS OF LAPLACE'S EQUATION

General case ($q = \alpha_3 c^2/4,\ a = \alpha_2 + \alpha_3 c^2/2$)

$$\frac{d^2H}{d\eta^2} - (a + 2q\cosh 2\eta)H = 0 \qquad H = ACe_m(\eta, -q) + BFe_m(\eta, -q),$$

or

$$H = ASe_m(\eta, -q) + BGe_m(\eta, -q).$$

$$\frac{d^2\Theta}{d\theta^2} + (a + 2q\cos 2\theta)\Theta = 0 \qquad \Theta = Ace_m(\theta, -q) + Bfe_m(\theta, -q),$$

or

$$\Theta = Ase_m(\theta, -q) + Bge_m(\theta, -q).$$

$$\frac{d^2Z}{dz^2} + \frac{4q}{c^2}Z = 0 \qquad Z = A\sin\left(\frac{2\sqrt{q}}{c}z\right) + B\cos\left(\frac{2\sqrt{q}}{c}z\right).$$

Special Solutions

$$H = Ae^{p\eta} + Be^{-p\eta}, \qquad \Theta = A\sin p\theta + B\cos p\theta.$$

$$\Theta = A + B\theta; \qquad H = A + B\eta.$$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case $(q = \alpha_3 c^2/4,\ a = \alpha_2 + \alpha_3 c^2/2)$

$$\frac{d^2H}{d\eta^2} - (a + 2q\cosh 2\eta)H = 0 \qquad H = ACe_m(\eta, -q) + BFe_m(\eta, -q).$$

$$\frac{d^2\Theta}{d\theta^2} + (a + 2q\cos 2\theta)\Theta = 0 \qquad \Theta = Ace_m(\theta, -q) + Bfe_m(\theta, -q),$$

$$\frac{d^2Z}{dz^2} + \left(\gamma^2 + \frac{4q}{c^2}\right)Z = 0 \qquad Z = A\sin\left(\sqrt{\gamma^2 + \frac{4q}{c^2}}\,z\right) + B\cos\left(\sqrt{\gamma^2 + \frac{4q}{c^2}}\,z\right).$$

Special Solutions

$$H = A + B\eta,\ \Theta = A + B\theta,\ Z = A\sin\gamma z + B\cos\gamma z.$$

TABLE IV.—*Parabolic-Cylinder Coordinates.*

$$u^1 = \mu,\ u^2 = \nu,\ u^3 = z$$

$$S = \begin{vmatrix} 0 & -1 & -\mu^2 \\ 0 & 1 & -\nu^2 \\ 1 & 0 & 1 \end{vmatrix} \qquad \begin{aligned} x^1 &= 1/2(\mu^2 - \nu^2) \\ x^2 &= \mu\nu \\ x^3 &= z \end{aligned}$$

$$f_1 = f_2 = f_3 = 1$$

$$g_{11} = g_{22} = \mu^2 + \nu^2,\ g_{33} = 1,\ \sqrt{g} = \mu^2 + \nu^2$$

$$(ds)^2 = [\mu^2 + \nu^2][(d\mu)^2 + (d\nu)^2] + (dz)^2$$

$$\nabla\phi = \frac{1}{\sqrt{\mu^2 + \nu^2}}\left[\mathbf{a}_\mu\frac{\partial\phi}{\partial\mu} + \mathbf{a}_\nu\frac{\partial\phi}{\partial\nu}\right] + \mathbf{a}_z\frac{\partial\phi}{\partial z}$$

$$\nabla^2\phi = \frac{1}{\mu^2 + \nu^2}\left[\frac{\partial^2\phi}{\partial\mu^2} + \frac{\partial^2\phi}{\partial\nu^2}\right] + \frac{\partial^2\phi}{\partial z^2}$$

$$\operatorname{div}\mathbf{E} = \frac{1}{\mu^2 + \nu^2}\left[\frac{\partial}{\partial\mu}(\sqrt{\mu^2 + \nu^2}\,E_\mu) + \frac{\partial}{\partial\nu}(\sqrt{\mu^2 + \nu^2}\,E_\nu)\right] + \frac{\partial E_z}{\partial z}.$$

$$\begin{aligned} \operatorname{curl}\mathbf{E} = {} & \mathbf{a}_\mu\left[\frac{1}{\sqrt{\mu^2 + \nu^2}}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] \\ & + \mathbf{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \frac{1}{\sqrt{\mu^2 + \nu^2}}\frac{\partial E_z}{\partial\mu}\right] \\ & + \frac{\mathbf{a}_z}{\mu^2 + \nu^2}\left[\frac{\partial}{\partial\mu}(\sqrt{\mu^2 + \nu^2}\,E_\nu) - \frac{\partial}{\partial\nu}(\sqrt{\mu^2 + \nu^2}\,E_\mu)\right]. \end{aligned}$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $\left(\alpha_2 = -q^2(p + \tfrac{1}{2}),\ \alpha_3 = \dfrac{q^4}{4}\right)$

$$\frac{d^2M}{d\mu^2} + \left[q^2(p + \tfrac{1}{2}) - \frac{q^4\mu^2}{4}\right]M = 0 \qquad M = AD_p(q\mu) + BD_{-p-1}(iq\mu).$$

$$\frac{d^2N}{d\nu^2} - \left[q^2(p + \tfrac{1}{2}) + \frac{q^4\nu^2}{4}\right]N = 0 \qquad N = AD_p(iq\nu) + BD_{-p-1}(q\nu).$$

$$\frac{d^2Z}{dz^2} + \frac{q^4Z}{4} = 0 \qquad Z = A\sin\left(\frac{q^2z}{2}\right) + B\cos\left(\frac{q^2z}{2}\right).$$

Special Solutions

$$M = A\sqrt{\mu}\, J_{1/4}\left(\frac{q\mu^2}{2}\right) + B\sqrt{\mu}\, N_{1/4}\left(\frac{q\mu^2}{2}\right),$$

$$N = A\sqrt{\nu}\, J_{1/4}\left(\frac{q\nu^2}{2}\right) + B\sqrt{\nu}\, N_{1/4}\left(\frac{q\nu^2}{2}\right),\ Z = Ae^{qz} + Be^{-qz}.$$

$$M = Ae^{p\mu} + Be^{-p\mu},\ N = A\sin p\nu + B\cos p\nu.$$

$$N = A + B\nu;\ M = A + B\mu.$$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case $\left(\alpha_2 = -\,q^2(p + \tfrac{1}{2}),\ \alpha_3 = q^4/4\right)$

$$\frac{d^2M}{d\mu^2} + \left[q^2(p + \tfrac{1}{2}) - \frac{q^4\mu^2}{4}\right] M = 0 \qquad M = AD_p(q\mu) + BD_{-p-1}(iq\mu).$$

$$\frac{d^2N}{d\nu^2} - \left[q^2(p + \tfrac{1}{2}) + \frac{q^4\nu^2}{4}\right] N = 0 \qquad N = AD_p(iq\nu) + BD_{-p-1}(q\nu).$$

$$\frac{d^2Z}{dz^2} + \left[\gamma^2 + \frac{q^4}{4}\right] Z = 0 \qquad Z = A\sin\left(\sqrt{\gamma^2 + \frac{q^4}{4}}\, z\right) + B\cos\left(\sqrt{\gamma^2 + \frac{q^4}{4}}\, z\right).$$

Special Solutions

$$M = A\sqrt{\mu}\, J_{1/4}\left(\frac{\gamma\mu^2}{2}\right) + BN_{1/4}\left(\frac{\gamma\mu^2}{2}\right),\ N = A\sqrt{\nu}\, J_{1/4}\left(\frac{\gamma\nu^2}{2}\right) + B\sqrt{\nu}\, N_{1/4}\left(\frac{\gamma\nu^2}{2}\right).$$

TABLE V.—*Spherical Coordinates.*

$$u^1 = r,\ u^2 = \theta,\ u^3 = \psi$$

$$S = \begin{vmatrix} 1 & -\dfrac{1}{r^2} & 0 \\ 0 & 1 & -\dfrac{1}{\sin^2\theta} \\ 0 & 0 & 1 \end{vmatrix}$$

$$x^1 = r\sin\theta\cos\psi$$

$$x^2 = r\sin\theta\sin\psi$$

$$x^3 = r\cos\theta$$

$$f_1 = r^2,\ f_2 = \sin\theta,\ f_3 = 1$$

$$g_{11} = 1,\ g_{22} = r^2,\ g_{33} = r^2\sin^2\theta,\ \sqrt{g} = r^2\sin\theta$$

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\psi)^2$$

$$\nabla\phi = \mathbf{a}_r\frac{\partial\phi}{\partial r} + \frac{\mathbf{a}_\theta}{r}\frac{\partial\phi}{\partial\theta} + \frac{\mathbf{a}_\psi}{r\sin\theta}\frac{\partial\phi}{\partial\psi}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}$$

$$\operatorname{div}\mathbf{E} = \frac{\partial E_r}{\partial r} + \frac{2E_r}{r} + \frac{1}{r}\frac{\partial E_\theta}{\partial\theta} + \frac{E_\theta}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial E_\psi}{\partial\psi}$$

$$\operatorname{curl}\mathbf{E} = \mathbf{a}_r\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\psi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial E_\theta}{\partial\psi}\right]$$

$$+ \mathbf{a}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial E_r}{\partial\psi} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\psi)\right]$$

$$+ \mathbf{a}_\psi\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial\theta}\right].$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $\left(\alpha_2 = p(p+1),\ \alpha_3 = q^2\right)$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{p(p+1)}{r^2}R = 0 \qquad R = Ar^p + Br^{-p-1}$$

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left[p(p+1) - \frac{q^2}{1-\mu^2}\right]\Theta = 0$$

$$\text{(where } \mu = \cos\theta) \quad , \quad \Theta = AP_p{}^q(\cos\theta) + BQ_p{}^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0 \qquad \Psi = A\sin q\psi + B\cos q\psi.$$

Special Solutions

$$R = Ar^p + Br^{-p-1}, \qquad \Theta = AP_p(\cos\theta) + BQ_p(\cos\theta)$$

$$R = A + Br^{-1}; \qquad \Theta = A + B\ln\cot\left(\frac{\theta}{2}\right).$$

SOLUTIONS OF THE HELMHOLTZ EQUATION $\left(\alpha_2 = p(p+1),\ \alpha_3 = q^2\right)$

General Case

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\gamma^2 - \frac{p(p+1)}{r^2}\right]R = 0 \qquad R = \frac{1}{\sqrt{r}}\left[AJ_{p+1/2}(\gamma r) + BN_{p+1/2}(\gamma r)\right].$$

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left[p(p+1) - \frac{q^2}{1-\mu^2}\right]\Theta = 0$$

$$\text{(where } \mu = \cos\theta) \qquad \Theta = AP_p{}^q(\cos\theta) + BQ_p{}^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0 \qquad \Psi = A\sin q\psi + B\cos q\psi.$$

Special Solutions

$$R = \frac{1}{\sqrt{r}}\left[AJ_{p+1/2}(\gamma r) + BN_{p+1/2}(\gamma r)\right], \qquad \Theta = AP_p(\cos\theta) + BQ_p(\cos\theta).$$

$$R = \frac{1}{r}(A\sin\gamma r + B\cos\gamma r).$$

TABLE VI.—*Prolate Spheroidal Coordinates.*

$$u^1 = \eta,\ u^2 = \theta,\ u^3 = \psi$$

$$S = \begin{vmatrix} c^2\sinh^2\eta & -1 & -\dfrac{1}{\sinh^2\eta} \\ c^2\sin^2\theta & 1 & -\dfrac{1}{\sin^2\theta} \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{aligned} x^1 &= c\sinh\eta\sin\theta\cos\psi \\ x^2 &= c\sinh\eta\sin\theta\sin\psi \\ x^3 &= \cosh\eta\cos\theta \end{aligned}$$

$$f_1 = \sinh\eta,\ f_2 = \sin\theta,\ f_3 = c$$

$$g_{11} = g_{22} = c^2(\sinh^2\eta + \sin^2\theta),$$

$$g_{33} = c^2\sinh^2\eta\sin^2\theta,\ \sqrt{g} = c^3\sinh\eta\sin\theta\,(\sinh^2\eta + \sin^2\theta)$$

$$(ds)^2 = c^2[\sinh^2\eta + \sin^2\theta][(d\eta)^2 + (d\theta)^2] + c^2\sinh^2\eta\sin^2\theta(d\psi)^2$$

$$\nabla\phi = \frac{1}{c\sqrt{\sinh^2\eta + \sin^2\theta}}\left[\mathbf{a}_\eta\frac{\partial\phi}{\partial\eta} + \mathbf{a}_\theta\frac{\partial\phi}{\partial\theta}\right] + \frac{\mathbf{a}_\psi}{c\sinh\eta\sin\theta}\frac{\partial\phi}{\partial\psi}$$

V.
19

$$\nabla^2\phi = \frac{1}{c^2(\sinh^2\eta + \sin^2\theta)}\left[\frac{\partial^2\phi}{\partial\eta^2} + \coth\eta\,\frac{\partial\phi}{\partial\eta} + \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\,\frac{\partial\phi}{\partial\theta}\right] + \frac{1}{c^2\sinh^2\eta\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}$$

$$\begin{aligned}\operatorname{div}\mathbf{E} = \frac{1}{c^3\sinh\eta\sin\theta(\sinh^2\eta + \sin^2\theta)}&\left[\frac{\partial}{\partial\eta}\left(c^2\sinh\eta\sin\theta\sqrt{\sinh^2\eta + \sin^2\theta}\,E_\eta\right)\right.\\ &\left.+ \frac{\partial}{\partial\theta}\left(c^2\sinh\eta\sin\theta\sqrt{\sinh^2\eta + \sin^2\theta}\,E_\theta\right)\right]\\ &+ \frac{1}{c\sinh\eta\sin\theta}\frac{\partial}{\partial\psi}E_\psi\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\mathbf{E} = \mathbf{a}_\eta&\left[\frac{1}{c^2\sinh\eta\sin\theta\sqrt{\sinh^2\eta + \sin^2\theta}}\frac{\partial}{\partial\theta}(c\sinh\eta\sin\theta E_\psi) - \frac{1}{c\sinh\eta\sin\theta}\frac{\partial}{\partial\psi}E_\theta\right]\\ &+ \mathbf{a}_\theta\left[\frac{1}{c\sinh\eta\sin\theta}\frac{\partial E_\eta}{\partial\psi} - \frac{1\,\frac{\partial}{\partial\eta}(c\sinh\eta\sin\theta\,E_\psi)}{c^2\sinh\eta\sin\theta\sqrt{\sinh^2\eta + \sin^2\theta}}\right]\\ &+ \frac{\mathbf{a}_\psi}{c^2(\sinh^2\eta + \sin^2\theta)}\left[\frac{\partial}{\partial\eta}\left(c\sqrt{\sinh^2\eta + \sin^2\theta}\,E_\theta\right) - \frac{\partial}{\partial\theta}\left(c\sqrt{\sinh^2\eta + \sin^2\theta}\,E_\eta\right)\right].\end{aligned}$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $\left(\alpha_2 = p(p+1),\ \alpha_3 = q^2\right)$

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2 - 1}\right]H = 0$$

(where $\xi = \cosh\eta$) $\quad H = AP_p{}^q(\cosh\eta) + BQ_p{}^q(\cosh\eta).$

$$(\rho^2 - 1)\frac{d^2\Theta}{d\rho^2} + 2\rho\frac{d\Theta}{d\rho} - \left[p(p+1) + \frac{q^2}{\rho^2 - 1}\right]\Theta = 0$$

(where $\rho = \cos\theta$) $\quad \Theta = AP_p{}^q(\cos\theta) + BQ_p{}^q(\cos\theta).$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0 \qquad \Psi = A\sin q\psi + B\cos q\psi.$$

Special Solutions

$$H = AP_p(\cosh\eta) + BQ_p(\cosh\eta), \qquad \Theta = AP_p(\cos\theta) + BQ_p(\cos\theta).$$

$$H = A + B\ln\coth(\eta/2); \qquad \Theta = A + B\ln\cot\left(\frac{\theta}{2}\right).$$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case $\left(\alpha_2 = p(p+1,\ (\alpha_3 = q^2\right)$

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{2\xi} + \left[\gamma^2c^2(\xi^2 - 1) - p(p+1) - \frac{q^2}{\xi^2 - 1}\right]H = 0$$

(where $\xi = \cosh\eta$) $\quad H = AP_p{}^q(\gamma c, \cosh\eta) + BQ_p{}^q(\gamma c, \cosh\eta).$

$$(\rho^2 - 1)\frac{d^2\Theta}{d\rho^2} + 2\rho\frac{d\Theta}{d\rho} + \left[\gamma^2c^2(\rho^2 - 1) - p(p+1) - \frac{q^2}{\rho^2 - 1}\right]\Theta = 0$$

where $\rho = \cos\theta \quad \Theta = AP_p{}^q(\gamma c, \cos\theta) + BQ_p{}^q(\gamma c, \cos\theta).$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0 \qquad \Psi = A\sin q\psi + B\cos q\psi.$$

Special Solutions

$$H = AP_p(\gamma c, \cosh \eta) + BQ_p(\gamma c, \cosh \eta),\ \Theta = AP_p(\gamma c, \cos \theta) + BQ_p(\gamma c, \cos \theta).$$

TABLE VII.—*Oblate Spheroidal Coordinates.*

$$u^1 = \eta,\ u^2 = \theta,\ u^3 = \psi$$

$$S = \begin{vmatrix} c^2 \cosh^2 \eta & -1 & \dfrac{1}{\cosh^2 \eta} \\ -c^2 \sin^2 \theta & +1 & -\dfrac{1}{\sin^2 \theta} \\ 0 & 0 & 1 \end{vmatrix}$$

$$x^1 = c \cosh \eta \sin \theta \cos \psi$$
$$x^2 = c \cosh \eta \sin \theta \sin \psi$$
$$x^3 = c \sinh \eta \cos \theta$$

$$f_1 = \cosh \eta,\ f_2 = \sin \theta,\ f_3 = c$$

$$g_{11} = g_{22} = c^2(\cosh^2 \eta - \sin^2 \theta)$$

$$g_{33} = c^2 \cosh^2 \eta \sin^2 \theta,\ \sqrt{g} = c^3 \cosh \eta \sin \theta(\cosh^2 \eta - \sin^2 \theta)$$

$$(ds)^2 = c^2(\cosh^2 \eta - \sin^2 \theta)[(d\eta)^2 + (d\theta)^2] + c^2 \cosh^2 \eta \sin^2 \theta (d\psi)^2$$

$$\nabla \phi = \frac{1}{c\sqrt{\cosh^2 \eta - \sin^2 \theta}} \left[\mathbf{a}_\eta \frac{\partial \phi}{\partial \eta} + \mathbf{a}_\theta \frac{\partial \phi}{\partial \theta} \right] + \frac{\mathbf{a}_\psi}{c \cosh \eta \sin \theta} \frac{\partial \phi}{\partial \psi}$$

$$\nabla^2 \phi = \frac{1}{c^2(\cosh^2 \eta - \sin^2 \theta)} \left[\frac{\partial^2 \phi}{\partial \eta^2} + \tanh \eta \frac{\partial \phi}{\partial \eta} + \frac{\partial^2 \phi}{\partial \theta^2} + \cot \theta \frac{\partial \phi}{\partial \theta} \right] + \frac{1}{c^2 \cosh^2 \eta \sin^2 \theta} \frac{\partial^2 \phi}{\partial \psi^2}$$

$$\begin{aligned} \operatorname{div} \mathbf{E} = \frac{1}{c^3 \cosh \eta \sin \theta (\cosh^2 \eta - \sin^2 \theta)} \Bigg[& \frac{\partial}{\partial \eta} \left(c^2 \cosh \eta \sin \theta \sqrt{\cosh^2 \eta - \sin^2 \theta}\, E_\eta\right) \\ & + \frac{\partial}{\partial \theta} \left(c^2 \cosh \eta \sin \theta \sqrt{\cosh^2 \eta - \sin^2 \theta}\, E_\theta\right) \Bigg] \\ & + \frac{1}{c \cosh \eta \sin \theta} \frac{\partial}{\partial \psi} E_\psi. \end{aligned}$$

$$\begin{aligned} \operatorname{curl} \mathbf{E} = {} & \mathbf{a}_\eta \left[\frac{1}{c^2 \cosh \eta \sin \theta \sqrt{\cosh^2 \eta - \sin^2 \theta}} \frac{\partial}{\partial \theta} (c \cosh \eta \sin \theta\, E_\psi) - \frac{1}{c \cosh \eta \sin \theta} \frac{\partial}{\partial \psi} E_\theta \right] \\ & + \mathbf{a}_\theta \left[\frac{1}{c \cosh \eta \sin \theta} \frac{\partial E_\eta}{\partial \psi} - \frac{1}{c^2 \cosh \eta \sin \theta \sqrt{\cosh^2 \eta - \sin^2 \theta}} \frac{\partial}{\partial \eta} (\cosh \eta \sin \theta\, E_\psi) \right] \\ & + \frac{\mathbf{a}_\psi}{c^2(\cosh^2 \eta - \sin^2 \theta)} \left[\frac{\partial}{\partial \eta} \left(c\sqrt{\cosh^2 \eta - \sin^2 \theta}\, E_\theta\right) - \frac{\partial}{\partial \theta} \left(c\sqrt{\cosh^2 \eta - \sin^2 \theta}\, E_\eta\right) \right]. \end{aligned}$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $\left(\alpha_2 = p(p + 1),\ \alpha_3 = q^2\right)$

$$(\xi^2 - 1) \frac{d^2 H}{d\xi^2} + 2\xi \frac{dH}{d\xi} - \left[p(p + 1) + \frac{q^2}{\xi^2 - 1} \right] H = 0$$

(where $\xi = i \sinh \eta$) $\quad H = AP_p{}^q(i \sinh \eta) + BQ_p{}^q(i \sinh \eta).$

$$(\rho^2 - 1) \frac{d^2 \Theta}{d\rho^2} + 2\rho \frac{d\Theta}{d\rho} - \left[p(p + 1) + \frac{q^2}{\rho^2 - 1} \right] \Theta = 0$$

(where $\rho = \cos \theta$) $\quad \Theta = AP_p{}^q(\cos \theta) + BQ_p{}^q(\cos \theta).$

$$\frac{d^2 \Psi}{d\psi^2} + q^2 \Psi = 0 \qquad \Psi = A \sin q\psi + B \cos q\psi.$$

Special Solutions

$$H = AP_p(i \sinh \eta) + BQ_p(i \sinh \eta), \qquad \Theta = AP_p(\cos \theta) + BQ_p(\cos \theta)$$

$$H = A + B \tan^{-1} (\sinh \eta); \qquad \Theta = A + B \ln \cot \left(\frac{\theta}{2}\right).$$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case $\left(\alpha_2 = p(p+1),\ \alpha_3 = q^2\right)$

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[\gamma^2c^2(\xi^2 - 1) + p(p+1) + \frac{q^2}{\xi^2 - 1}\right]H = 0$$

(where $\xi = i \sinh \eta$) $\quad H = AP_p{}^q(i\gamma c, i\sinh\eta) + BQ_p{}^q(i\gamma c, i\sinh\eta)$

$$(\rho^2 - 1)\frac{d^2\Theta}{d\rho^2} + 2\rho\frac{d\Theta}{d\rho} - \left[\gamma^2c^2(\rho^2 - 1) + p(p+1) + \frac{q^2}{\rho^2 - 1}\right]\Theta = 0$$

(where $\rho = \cos\theta$) $\quad \Theta = AP_p{}^q(i\gamma c, \cos\theta) + BQ_p{}^q(i\gamma c, \cos\theta).$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0 \qquad \Psi = A\sin q\psi + B\cos q\psi.$$

Special Solutions

$$H = AP_p(i\gamma c, i\sinh\eta) + BQ_p(i\gamma c, i\sinh\eta),\ \Theta = AP_p(i\gamma c, \cos\theta) + BQ_p(i\gamma c, \cos\theta).$$

TABLE VIII.—*Parabolic Coordinates.*

$$u^1 = \mu,\ u^2 = \nu,\ u^3 = \psi$$

$$S = \begin{vmatrix} \mu^2 & -1 & -\frac{1}{\mu^2} \\ \nu^2 & 1 & -\frac{1}{\nu^2} \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{aligned} x^1 &= \mu\nu\cos\psi \\ x^2 &= \mu\nu\sin\psi \\ x^3 &= \tfrac{1}{2}(\mu^2 - \nu^2) \end{aligned}$$

$$f_1 = \mu,\ f_2 = \nu,\ f_3 = 1$$

$$g_{11} = g_{22} = \mu^2 + \nu^2,\ g_{33} = \mu^2\nu^2,\ \sqrt{g} = \mu\nu(\mu^2 + \nu^2)$$

$$(ds)^2 = (\mu^2 + \nu^2)[(d\mu)^2 + (d\nu)^2] + \mu^2\nu^2(d\psi)^2$$

$$\nabla\phi = \frac{1}{\sqrt{\mu^2 + \nu^2}}\left[\mathbf{a}_\mu\frac{\partial\phi}{\partial\mu} + \mathbf{a}_\nu\frac{\partial\phi}{\partial\nu}\right] + \frac{\mathbf{a}_\psi}{\mu\nu}\frac{\partial\phi}{\partial\psi}$$

$$\nabla^2\phi = \frac{1}{\mu^2 + \nu^2}\left[\frac{\partial^2\phi}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial\phi}{\partial\mu} + \frac{\partial^2\phi}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial\phi}{\partial\nu}\right] + \frac{1}{\mu^2\nu^2}\frac{\partial^2\phi}{\partial\psi^2}$$

$$\operatorname{div}\mathbf{E} = \frac{1}{\mu\nu(\mu^2 + \nu^2)}\left[\frac{\partial}{\partial\mu}\left(\mu\nu\sqrt{\mu^2 + \nu^2}\,E_\mu\right) + \frac{\partial}{\partial\nu}\left(\mu\nu\sqrt{\mu^2 + \nu^2}\,E_\nu\right)\right] + \frac{1}{\mu\nu}\frac{\partial}{\partial\psi}E_\psi$$

$$\begin{aligned}\operatorname{curl}\mathbf{E} = \mathbf{a}_\mu &\left[\frac{1}{\mu\nu\sqrt{\mu^2 + \nu^2}}\frac{\partial}{\partial\nu}(\mu\nu E_\psi) - \frac{1}{\mu\nu}\frac{\partial}{\partial\psi}E_\nu\right] \\ &+ \mathbf{a}_\nu\left[\frac{1}{\mu\nu}\frac{\partial}{\partial\psi}E_\mu - \frac{1}{\mu\nu\sqrt{\mu^2 + \nu^2}}\frac{\partial}{\partial\mu}(\mu\nu E_\psi)\right] \\ &+ \frac{\mathbf{a}_\psi}{\mu^2 + \nu^2}\left[\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2 + \nu^2}\,E_\nu\right) - \frac{\partial}{\partial\nu}\left(\sqrt{\mu^2 + \nu^2}\,E_\mu\right)\right].\end{aligned}$$

SOLUTIONS OF LAPLACE'S EQUATION

General case $(\alpha_2 = q^2,\ \alpha_3 = p^2)$

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - \left(q^2 + \frac{p^2}{\mu^2}\right)M = 0 \qquad M = AJ_p(iq\mu) + BN_p(iq\mu)$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + \left(q^2\frac{p^2}{\nu^2}\right)N = 0 \qquad N = AJ_p(q\nu) + BN_p(q\nu).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0 \qquad \Psi = A\sin p\psi + B\cos p\psi.$$

Special Solutions

$$M = AJ_0(iq\mu) + BN_0(iq\mu), \quad N = AJ_0(q\nu) + BN_0(q\nu).$$

$$M = A + B \ln \mu, \qquad N = A + B \ln \nu.$$

SOLUTIONS OF THE HELMHOLTZ EQUATION

General case ($\alpha_2 = q^2, \alpha_3 = p^2$)

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + \left(\gamma^2\mu^2 - q^2 - \frac{p^2}{\mu^2}\right)M = 0 \qquad M = AJ_p(\gamma, iq\mu) + BN_p(\gamma, iq\mu) \text{ or}$$

$$M = \mu^p e^{\pm i\gamma\mu^2/2} {}_1F_1\left(\frac{iq^2}{4\gamma} + \frac{p+1}{2}; p+1; \mp i\gamma\mu^2\right)$$

$$\text{and } M = \mu^{-1}W_{iq^2/4\gamma,\, p/2}(i\gamma\mu^2).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + \left(\gamma^2\nu^2 + q^2 - \frac{p^2}{\nu^2}\right)N = 0 \qquad N = AJ_p(\gamma, q\nu) + BN_p(\gamma, q\nu), \text{ or}$$

$$N = \nu^p e^{\pm i\gamma\nu^2/2} {}_1F_1\left(\frac{-iq^2}{4\gamma} + \frac{p+1}{2}; p+1; \mp i\gamma\nu^2\right)$$

$$\text{and } N = \nu^{-1}W_{q^2/4i\gamma,\, p/2}(i\gamma\nu^2).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0 \qquad \Psi = A \sin p\psi + B \cos p\psi.$$

Special Solutions

$$M = AJ_0(\gamma, iq\mu) + BN_0(\gamma, iq\mu), \quad N = AJ_0(\gamma, q\nu) + BN_0(\gamma, q\nu).$$

NOTE ON "OPERATIONAL APPROACH TO NON-LINEAR CIRCUIT ANALYSIS"*

BY

GERALD H. COHEN

Letter to the Editor:

Since publication of "Operational Approach to Non-Linear Circuit Analysis" (June, 1949) an attempt has been made to treat rigorously the problem of a non-linear circuit. It now appears that the heuristic method used leaves considerable doubt as to the validity of the conclusions.

It was brought to my attention that C. S. Roys[1] previously treated the same problem using operational calculus. It appears that his method is more valid since there is less ambiguity concerning the use of complex operators.

It should also be pointed out that Eq. 13 of my paper was in error. The transform of $E^2 \sin^2 \omega t$ should have been written

$$\frac{E^2}{2}\left[\frac{1}{s} - \frac{s}{s^2 + (2\omega)^2}\right]$$

4/19/51 GERALD H. COHEN

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* This JOURNAL, Vol. 247, p. 573 (1949).

[1] C. S. ROYS, "The Transient Behavior of Non-Linear Systems," *Proceedings Nat. Electronics Conf.*, Vol. 3, pp. 663–669 (1947).

NOTES FROM THE NATIONAL BUREAU OF STANDARDS*

AN AIR-LAUNCHED AUTOMATIC WEATHER STATION

A self-contained automatic weather station, which transmits weather data by radio, has been developed by Percival D. Lowell and William Hakkarinen of the National Bureau of Standards for the Navy Bureau of Ships. Named the "Grasshopper," the device can be parachuted from aircraft onto inaccessible territory. Developed during World War II, it will automatically set itself up and periodically make and transmit weather observations. It may also be used as a radio marker beacon.

After the station has parachuted to earth, controlled explosive charges are used to disengage the parachute, raise the station to an upright operating position, and erect a telescoping antenna. Weather-responsive devices then cause resistance changes which switch a radio transmitter on and off at a rate susceptible of translation by a receiving station into temperature, pressure, and humidity readings.

Designed in the shape of a bomb, and packing its own parachute, the weather station is loaded on the bomb rack of an aircraft. When the unit is released over a desired location, the parachute is automatically opened by a line rigged from the aircraft. Simultaneously, an electric clock, which controls subsequent operations of the station, is turned on. The impact of landing sets off a small explosive charge which disengages the parachute and prevents the station from being pulled along the ground. Either immediately or after a pre-set dormancy period, another explosive charge causes the station to rise to an upright operating position. This is done through an arrangement of six legs to which springs are attached; the explosive charge operates a release, permitting the springs to pull the legs into position. A third explosive charge extends a telescopic vertical antenna to a height of some 20 feet. The station is then ready for automatic transmission at intervals predetermined by the built-in timing mechanism.

The automatic station could be adapted to transmit various kinds of information, but in the standard design only temperature, pressure, and humidity data are reported. Separate mechanisms responsive to changes in these atmospheric conditions each cause an associated resistor to vary. At predetermined intervals the timing mechanism turns on the radio transmitter and connects one resistor after another to a critical point in the transmitter circuit. The transmitter is designed so that the emitted radio signal pulses on and off at a rate proportional to the value of the resistor so connected. The station is calibrated before use by subjecting it to known temperatures, pressures, and humidities and measuring the resulting pulse rates. At the receiving station the transmitter pulse rate can then be read as temperature, pressure or humidity, depending on the phase of the predetermined clockwork cycle.

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* Communicated by the Director.

The radio transmitter proper consists of a crystal oscillator followed by a radio-frequency amplifier stage. A relay in the plate circuit of a separate relaxation oscillator turns the crystal oscillator on and off at a rate proportional to the value of whatever resistor is temporarily inserted (by the clock mechanism) into the relaxation oscillator circuit. When the station is to be used as a beacon, the radio transmitter and its control mechanisms may, of course, be simplified.

The clock, in addition to inserting the several weather-responsive resistors into the circuit in a predetermined sequence, connects two other resistors at appropriate intervals. These are a reference resistor and an identification resistor, both of constant value. The pulse rate produced by the fixed reference resistor is observed during initial tests of the transmitter. Any subsequent deviation in the reference resistor pulse rate warns the receiving station that a correction factor must be applied to the pulse rates of the weather-responsive resistors. Such deviation could arise from transmitter damage or aging. The identification resistor is of a value selected to produce a pulse rate characteristic of the particular station; this enables the receiving station operator to identify the station.

A special technique is used to insure maximum accuracy of the transmitted data despite possible deformation of the weather-responsive mechanisms due to landing impact: a buzzer vibrates each weather-responsive device for a short time before its associated resistor is inserted in the relaxation oscillator circuit. This forced vibration counteracts friction, which may have been increased by landing-impact deformation, and thus aids in the attainment of a true equilibrium condition.

The developmental model of the weather station had an output of the order of 5 watts. Operating on a frequency in the neighborhood of 5 megacycles, it performed reliably over land at ranges of more than 100 miles. The dry batteries used provided power for transmission of weather reports at 3-hr. intervals for more than 15 days.

MOBILE IONOSPHERIC STATION

The propagation of radio waves over long distances depends on their reflections from the ionosphere, a series of electrically conducting layers in the earth's atmosphere. Because these layers are continuously changing, knowledge of their characteristics is necessary for regular and reliable radio communications. The Central Radio Propagation

Laboratory of the National Bureau of Standards collects and analyzes ionospheric data from stations all over the world. From the data, the Bureau makes and publishes predictions of usable frequencies for communications between any two places at any hour.

In an effort to obtain more comprehensive data on the ionosphere, the Bureau has now incorporated a mobile ionospheric research unit into the existing chain of 60 world-wide ionosphere stations—approximately one quarter of which are maintained by the Bureau. With this new unit, ionospheric soundings will be made from points midway between two permanent transmitting-receiving stations. This information will be used in studies of ionospheric effects on radio waves directly above the mobile transmitter-receiver; and it will also aid in analyzing the behavior of radio waves propagated from transmitters to distant receivers.

The NBS mobile unit might more properly be called an ionosphere research caravan because it is made up of two prime movers and two trailers. The equipment includes two gasoline-powered motor-generators which will provide 10,000 watts each of electrical power for the transmitting, receiving, and recording components in the event commercial power is unavailable. One of the trailers has been converted into living quarters for the two-man operating crew for use in regions where living accommodations cannot be obtained.

The caravan's first recording stop, 30 miles east of Cincinnati, Ohio, is a point midway between the Bureau's transmitting station at Sterling, Virginia, and a leased transmitting station operated by Washington University, St. Louis, Mo. The project, known as the "St. Louis Experiment," is expected to continue for three or four months. When the experiment is completed, the unit will be moved to another center-point site. Within a few years and after a number of midpoint locations, the Bureau expects to have sufficient information about the ionospheric radio propagation to map accurately the paths of radio waves across the country and into other parts of the world.

During the past 20 years the technique of vertical-incidence ionosphere soundings by the use of pulse transmitters scanning over a wide frequency range has been adopted internationally for recording the virtual height and degree of ionization for each of the ionosphere layers. Although vertical incidence reflection principles are basic to the understanding of radio propagation, they are rarely encountered in practical radio communication. The wave received at a distance from the transmitter impinges on the ionosphere obliquely in its journey from the transmitter to the receiver. However, theory indicates that much information about oblique incidence propagation can be inferred from data obtained at vertical incidence. The St. Louis experiment and all of those to follow are designed to record simultaneously both vertical and oblique incidence data, and from an analysis of the information the relation between oblique and vertical incidence reflectors will be more exactly deduced.

In the St. Louis experiment, the Sterling and St. Louis stations will simultaneously transmit pulses of radio energy directed to strike the ionosphere obliquely. The radio waves will be propagated in west and east directions respectively, and each station will receive the other's signals. Because the pulses must pass all the way up to the ionosphere at the midpoint of the path and then be reflected down again, the travel times will be greater than if the signals traveled directly along the earth's surface. From the time delays involved, the time of ionospheric travel between the stations will be obtained and the heights of reflection calculated. During the transmission, the radio frequencies used at both ends of the paths will be slowly but simultaneously varied. At the higher frequencies the time delays will be greater because these waves travel to greater altitudes before they are reflected. Meanwhile the mobile station located at the critical point of the propagation path, will transmit signals vertically upward and receive its own echoes. By recording the heights and degree of ionization of the several layers, the heights at which, according to simple theory, the oblique incidence signals should be reflected, can be calculated.

All three transmitting-receiving stations will use the model C-3 automatic ionosphere recorder developed by the Bureau. Its transmitted power is 20 kw. and it is capable of scanning a frequency range of 1 to 25 mc. in a time interval as small as 7.5 seconds. Signals will be transmitted, received, and recorded on film strips at a 24-hr. per day basis.

The "St. Louis Experiment" involves comparatively short distances. However, it is known that the discrepancies between theory and practice increase in magnitude as the distance between transmitter and receiver becomes greater. To investigate this effect, the mobile unit will be placed at the midpoint of much more widely spaced permanent installations. The next stop may be between the Sterling station and one in the Hawaiian Islands, Alaska, San Francisco, Puerto Rico, or Panama. Before the program is completed, the mobile unit will have been placed at the midpoint of the path to most of the Bureau's ionosphere stations in the western hemisphere.

The Bureau will analyze the film-strip records from each tri-station experiment. The analysis will shed more light on the causes of fading and explain the complete loss of a signal between the transmitter and the receiver. Thus, at the end of this unusual series of experiments, the Bureau will be in a position to supply all the users of long-range communications with more accurate information for selecting the proper frequency to transmit intelligence over a certain path at any particular time of the day and year.

ATOMIC STANDARDS OF LENGTH NOW AVAILABLE

The availability to science and industry of an ultimate standard of length was announced recently by the National Bureau of Standards and the Atomic Energy Commission. The standards consist of spectroscopic lamps containing a single pure isotope of mercury. These lamps enable any research organization which has the auxiliary optical equipment to have for the first time an ultimate primary standard of length in its own laboratories.

Distribution of the lamps will be handled by the National Bureau of Standards. They will be available to qualified government, industrial and educational laboratories both in this country and abroad, engaged in precision length measurements and related research. All requests for information and applications should be addressed to the Coordinator of Atomic Energy Commission Projects at the National Bureau of Standards, Washington 25, D. C.

The NBS-Meggers Mercury 198 Lamps, prepared under the direction of Dr. W. F. Meggers, Chief of the National Bureau of Standards Spectroscopy Laboratory, contain about one milligram of mercury of atomic weight 198. Mercury 198 is obtained by the transmutation of gold in a nuclear chain-reacting pile. In 1931, Dr. Meggers suggested that an ultimate standard of length could be obtained if one of the even atomic weight isotopes of mercury could be isolated from the seven which occur all mixed together in natural mercury. At that time no means existed for obtaining useful amounts of a single isotope. In 1945, following the war-time development of the nuclear pile, the production of a useful quantity of artificial mercury 198 was achieved at Oak Ridge by subjecting pure gold to intense neutron bombardment.

Mercury 198 is sealed in a small glass tube and caused to glow brightly by the application of high frequency radio waves. The light waves emitted by the single isotope have extremely sharply defined wavelengths which can be used for length measurements of great precision. These particular waves are uniquely characteristic of the mercury and remain unaltered with time.[1]

The new standard of length is based upon the wavelength of the green light emitted by the lamp. This wavelength, near 21 millionths of an inch, is consistently emitted with a reproducibility greater than one part in a billion. Length measurements based upon it can readily be made with an accuracy of one part in 100 million. New apparatus now under construction at the National Bureau of Standards is expected to extend the accuracy to one part in a billion.

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[1] For a more complete description of Hg 198, see "Light Wave of Mercury 198 as Ultimate Standard of Length," *NBS Technical News Bulletin*, Vol. 31, pp. 12, 133 (Dec. 1947); also "Lamps and Wavelengths of Mercury 198," by Wm. F. Meggers and F. Oliver Westfall, *J. Research NBS*, Vol. 44, p. 447 (1950), RP 2091.

The measurement of length in terms of a characteristic radiation from a particular atom involves the optical technique of interferometry whereby the separation of two precisely flat plates is measured in terms of the number of wavelengths of light contained in the distance between them. Because the number of waves contained in any practical distance is determinable without error and because the wavelength of the light from a single, pure isotope of even atomic weight is so sharply defined, accuracies of measurement heretofore impossible can be obtained.

Although the world's official primary standard of length is still the distance between two lines on a metal bar, practically all precise measurements of length in the 20th century have been made and will continue to be made with light waves. When accuracy greater than the part in 10 million possible with the standard meter is necessary, only the new spectroscopic standard meets the need.

Up to the present time the wavelength of the red light from cadmium has been used as a wavelength standard. In 1927 the International Conference of Weights and Measures provisionally adopted the value 1,553,163.13 wavelengths of cadmium red radiation as equal to 1 meter. In 1948 the NBS proposed that the meter be defined in terms of the radiation from mercury 198. This is now under consideration. Present measurements show that 1,831,249.21 wavelengths of the green radiation from mercury 198 equal 1 meter.

The fundamental advantage of mercury over cadmium is that the wavelengths of the light from mercury are much more sharply defined. Cadmium consists of six principal isotopes which are not separated. They emit light of slightly different wavelengths. When length measurements are made with light having this mixture of wavelengths, a region of confusion results which limits the accuracy. Mercury has other advantages: (1) the atoms are heavier, move about more slowly, and hence disturb the wavelength less; and (2) the light can be obtained without heating the mercury to a high temperature and thereby speeding up the atomic motion. The life of the NBS-Meggers Mercury 198 Lamps appears to be infinite; a lamp under continuous excitation since October 1949—over 10,000 hours—has shown no significant deterioration. When all the advantages are considered, it is found that the mercury 198 provides a standard 300% more accurate than cadmium.

Recently the Oak Ridge Laboratories of the Atomic Energy Commission have succeeded in isolating practical quantities of mercury 202 by magnetic separation methods. The advantages of this isotope over the 198 isotope are negligible except for certain important problems in atomic research. Mercury 202 lamps, if needed for research in atomic physics, can also be obtained from the National Bureau of Standards by special arrangement. The wavelength of the light from the 202 isotope has not as yet been accurately measured and hence the information needed for its use as an atomic length standard is not available.

THE FRANKLIN INSTITUTE

ANNUAL REPORT OF THE BOARD OF MANAGERS FOR THE YEAR 1950

The Annual Report of the Board of Managers for the calendar year 1950 is presented to the Institute's membership by the President in behalf of that Board, as follows:

Officers and Board of Managers

It is with profound sorrow that there must be recorded the losses through death of Mr. Samuel S. Fels, one of the Institute's Vice Presidents, and Mr. E. Paul du Pont, a member of the Board of Managers. Much regretted further losses occurred through the resignations of Mr. W. Chattin Wetherill, a Vice President, and Mr. William J. Meinel, a member of the Board of Managers. Mr. Edward G. Budd, Jr. and Mr. James H. Robins were elected to the vice presidential vacancies thus caused, and Mr. Wilfred D. Gillen, Mr. A. Felix duPont, and Dr. James Creese were elected to vacant memberships on the Board of Managers.

In July 1950, the Executive Vice President was requested by the Mayor of Philadelphia, as Coordinator of Civil Defense for Philadelphia County, to head the Technical Division of the Philadelphia County Civil Defense Council. The appointment was accepted after consultation with the President and the Chairman of the Executive Committee; all agreed that The Franklin Institute must be receptive to so important an opportunity for community service. Through the activities of the Bartol Foundation, and the Manager of the Laboratories' Administration Division, with the occasional or incidental assistance of other staff members, the Institute has contributed much in time and effort to the Civil Defense program.

Membership

The Membership Department began several innovations in 1950, including the awarding of gold keys and gold cards to those who had been members continuously for fifty years or more and the awarding of silver cards to those who had been members continuously for twenty-five years or more, up to fifty years.

Statistically, the membership totalled 5591 at the end of 1950, a net increase of 45 for the year. The election of 974 new members through the suggestions of our members was a major contribution to this result.

The "Science Is Fun" program, preceded by informal dinners, reached into the fields of Electricity and Frozen Foods, and included a preview of the Fels Planetarium show "Rocket to Mars."

Friends of Franklin

The group of public-spirited individuals and companies known as the Friends of Franklin continued its generous support of the Institute. The amount contributed by them in 1950 was $27,408.00 (of which $1965 was applied to membership dues). This amount included subscriptions from 162 new Friends gained as a result of the special campaign for the purpose.

Meetings

During the year 1950 the average attendance at stated meetings and at meetings held jointly with various technical societies was 259. The dinners preceding these meetings were attended by an average of 127 members and guests. The Edward G. Budd Lecture was given by William L. Batt, and the Charles Day Lecture by J. Howard Pew. The talks were all interesting and timely and apparently enjoyed by their audiences.

The Guy Marriner piano lecture-recitals continued to be popular, and reached the fourteenth series with their reopening, in October 1950.

The Institute's meeting rooms, and particularly its well-equipped Lecture Hall, are so attractive to other scientific organizations and technical societies that they are constantly in demand.

Committee on Science and the Arts

The entire Science and Arts Committee met eight times. Thirty-two Sub-Committee meetings were held. Eleven medals were awarded at the annual Medal Day Exercises, one of which was a Franklin Medal awarded to Dr. Eugene P. Wigner. Three George A. Hoadley Certificates were presented during the year as follows:

Mr. William G. Ellis—twenty-five years of service
Mr. Coleman Sellers, 3rd—twenty-five years of service
Mr. Walter C. Wagner—twenty-five years of service.

The Journal of The Franklin Institute

The December 1950 number of the JOURNAL marked the completion of 125 years of continuous publication—a record of which we may well be proud when there is judged not only the age of the vehicle but also the quality of the material it has carried. In 1950, plans were consummated for the publication of a special 125th Anniversary edition in January, 1951. Papers on the future prospects in various branches of science and technology were solicited for the special issue, from outstanding authorities in their fields. The editors are grateful to the eminent men who contributed so solidly to the success of the Institute's anniversary number.

During the year 74 manuscripts were submitted for consideration, of which eight were published and 45 were added to the waiting list. Through rigorous screening of the manuscripts, the accumulative backlog was reduced from a two-year to a sixteen-month supply of papers. The Publications Committee owes much to the men who give their services as reviewers, and wishes here to extend to them the thanks which they deserve.

An increase in revenue from subscriptions and from advertising for 1950 was more than offset by an increase in printing costs. It was deemed advisable, therefore, not to attempt to expand the JOURNAL beyond its present size of 112 pages.

During 1950 the JOURNAL published one general-interest paper each month, presenting scientific subjects in a form not too abstruse for the interested layman. This feature will be continued, since it provides our lay members with a nontechnical, yet more than "popular," approach to various scientific problems.

Library

On January 1, 1951, the library contained 136,657 bound volumes, 49,554 pamphlets and 6136 maps, being notably strong in periodicals, patents and books on chemistry, physics and engineering. During the year it served 9150 readers, handled 5576 telephone calls for information and supplied 9196 photostats of material from its shelves.

Acquisitions in 1950 totalled 4578 items exclusive of current periodicals and patents. Gifts included 597 bound volumes, 635 pamphlets and 815 maps. As the result of binding previously unaccessioned material 1508 volumes were added, and 1023 volumes were acquired by purchase. Most of the important English language books in our fields were purchased as soon as they were published and a special effort was made to catalogue and shelve them promptly. Book funds are, of course, finite, and it has not been possible to obtain as much of the foreign literature as could be wished. However, a selection of outstanding current French and Swiss books as well as some of the German works of the war period have been secured. Check lists of current acquisitions appear regularly in the JOURNAL OF THE FRANKLIN INSTITUTE. These lists, although not complete, have been useful to patrons of the Library and to librarians of other institutions, since they call attention to worthwhile new books in the fields of science and technology.

Following out a policy established in 1949, special attention was paid during 1950 to strengthening our holdings of bibliographies. We have continued to receive the numerous periodical publications devoted to abstracts of the current scientific and technical literature, have purchased all the pertinent separately printed bibliographies which appeared during the year, and have made some progress in acquiring the older bibliographies needed to cover our fields of interest.

The Library currently receives 1200 periodicals. They are in constant use and probably constitute the most valuable part of our reference collection. Many foreign periodicals were interrupted during the war. Continuous efforts are being made to complete our files of the more important European journals and progress during 1950 was substantial. Much remains to be done, however, for publishers' stocks in Leipzig and Berlin were destroyed by bombing and it has not yet been possible to secure copies of some of the German language publications issued from 1940 to 1945.

The Ware Sugar Collection is extraordinarily rich in books, pamphlets and serial publications on every aspect of sugar. It is being kept up to date by the acquisition of all newly published works within its scope. It has been further strengthened during 1950 by the purchase of a number of scarce 17th and 18th century pamphlets on the sugar industry of the West Indies and by a group of late 19th century manuscripts and ephemera from the files of Macfie & Son, Liverpool refiners. A bibliography of the important early books in the collection is being prepared for publication.

During 1950 we have added 65 rare books to the Library, which have measurably increased its permanent interest and value.

The Library has funds which enable it to acquire material in a way even approaching completeness only for the fields of Sugar, Mechanics and Manufacturing Gas. Funds are badly needed for books in the specific fields of Chemistry, Electricity and Physics, among others.

Museum and Planetarium

Attendance:

Comparative attendance was

	Paid	*Free*	*Total*
1949	167,844	76,588	244,432
1950	162,958	67,814	230,772

Students from the public and parochial schools of Philadelphia continued to be admitted without charge when attending in groups accompanied by teachers.

From records kept during the last quarter, it was found that the Museum visitors were drawn from every state in the Union and from 44 foreign countries.

Exhibits:

To the Museum's policy of displaying technical achievement in an entertaining and instructive manner, the Bell System made a notable contribution by providing a separate room in which is displayed a series of exhibits illustrating the telephone and the system of communication it makes possible. Numerous details which have led to the improvement of telephone communication are displayed in a way which clearly explains to the layman how the difficulties attending the conquest of space by human speech have been overcome. Courteous and efficient demonstrators are in continuous attendance. The ingenious manner in which the exhibits have been prepared (thirteen so that they can be operated by the visitor) makes this one of the most notable exhibits in this or any other museum.

The Hall of Electrical Communications which had to be disarranged to permit the construction of the Bell System room has been revised and restored to offer a better sequential presentation of the subject.

The second annual Greater Philadelphia Science Fair held in Franklin Hall with the co-sponsorship of the *Philadelphia Inquirer* justifies the belief that this feature of Museum activity will progressively develop in interest. This year 363 exhibits were submitted, an increase of more than one hundred per cent over the year before. The resultant growth of interest on the part of the students has been gratifying to ourselves and our co-sponsor. The success attending the exhibition encouraged *Science Service* to promote the first National Science

Fair. This was held in Franklin Hall from 19 to 21 May. It consisted of a display of the prize-winning entries of the local science fairs held throughout the country.

A valuable collection of United States postage stamps was bequeathed to The Franklin Institute by the late Mr. Harry Jefferys. This collection is being arranged by a group of experts so that the stamps will be exhibited in a manner which will render them convenient for examination by collectors. This collection will form a fitting tribute to Benjamin Franklin as the first Postmaster General of the United States.

From April 18 to 20, an instructive lecture with numerous demonstrations illustrating the "March of Research" was given in the Museum by Dr. Richard C. Hitchcock, of the Westinghouse Electric Corporation.

The fifth annual Book Show of Philadelphia was held from April 11 to 23. The exhibits covered most of the phases of Graphic Arts related to book publishing.

A model of an express type of roadway, prepared by the U.S. Bureau of Public Roads, was demonstrated during the month of April.

Because of the stringency in both exhibit space and storage space, great care must continue to be exercised in the choice of exhibits accepted for display. Among the thirty-one exhibits received during the year were:

Industrial temperature measurement methods from Leeds & Northrup.

Demonstration of petroleum and chemical firefighting with foam from the National Foam Company.

An exhibit illustrating the clean-up of the Schuylkill River from the Pennsylvania Department of Forests and Waters.

An oscillating headlight for locomotives from the Mars Signal Light Company.

Specimens of colloids from Dr. H. N. Holmes, Oberlin College.

McIntosh X-Ray machine (1918) from Dr. C. A. R. McClain, Mount Union, Pa.

Edison home phonograph from Mrs. P. H. Dannebauer, Ardmore, Pa.

Enlargements of the first photographs of the earth taken from an altitude of 70 miles, from Johns Hopkins University.

Special Activities:

For the sixth season the members of the Museum staff have cooperated with station WFIL–TV and the public, parochial, and private schools of the area in the presentation of "Science Is Fun," formerly a radio program, but now presented by television.

The experimental course of conducted tours through the Museum for young people, called "Science Adventures," having proved successful, a second series was carried out during the summer months with renewed success.

Most of the customary planetarium demonstrations were given during the course of the year with two notable additions. In February an entirely new show, "The Universe and Dr. Einstein," was devoted to the subject of relativity, the first time this type of demonstration had been offered in a planetarium. Many special effects were designed and built. The attendance was the highest recorded for any February month except that of 1934, three months after the Fels Planetarium was opened. In the spring and again in August, another new show, "Rocket to Mars," calling for new scenery and effects, was presented. The public response was so gratifying that this demonstration will be added to the permanent planetarium repertoire.

Personnel

The year 1950 was an active one for the Personnel Department. On January 1 there were 407 employees. During the year there were 130 separations and 132 additions, leaving 409 employees on December 31. There are recorded with regret the deaths of Julius A. Mayer, John D. Miller, and Charles A. Wilson.

April saw the start of the contributory pension plan and November the adoption of a job evaluation system.

The Nation's rearmament program began to be felt, bringing the added problems of increased personnel requirements, greater turnover, and draft deferments.

Hostess Committee

The Hostess Committee has continued to earn our gratitude for its interest in and its effective contributions to the Institute's needs. Continuation of the grounds-improvement program was the Committee's chief project for 1950.

Public Relations

During 1950, the Public Relations Department continued to extend and add to its activities. Continual cooperation with all other Institute departments was maintained in order better to inform the public regarding the Institute's aims, purposes and services.

The Institute News was published ten times during the year, two months being covered by a single issue twice during the summer.

New approaches to the public were adopted. Radio and television appearances were increased, as were speeches made before various groups. Local, state, and national newspapers and magazines kept up their support and cooperation.

Two exhibits were displayed in the foyer of a large, downtown Philadelphia department store.

Special previews were held for two planetarium demonstrations. The first Eastern United States showing of an unusual astronomical motion picture was held in conjunction with a third planetarium demonstration.

Groundwork was laid for a survey to be made by the Charles Morris Price School of Advertising.

Several hundred inquiries regarding the Institute or Benjamin Franklin were met by supplying pertinent information or photographs.

Building Operation and Maintenance

It has become timely to note that our building and its equipment have now been in use for more than seventeen years. The effects of wear are naturally being felt. While special attention to preventive maintenance has thus far been successful in preventing serious or expensive breakdowns, it is advisable to realize the potential effects of increasing age on the costs of future operations and maintenance.

Research Laboratories

The Bartol Research Foundation

The Bartol Foundation was engaged in work under the following heads:

1. Researches in nuclear physics and cosmic rays.
2. Researches in matters pertaining to thermionic work.
3. Researches on secondary electron emission from metals.
4. Work having to do with the development and construction of a new Van de Graaff generator for positive ions and for a voltage range between five and ten million volts.
5. Development of a small linear accelerator.
6. Miscellaneous researches sponsored by the Bartol Foundation itself.

The new laboratory which houses the large generator now under construction was completed and dedicated. Additional land was purchased for possible future expansion.

All of the major parts of the generator have been constructed and the generator itself has been built up to one-third of its final height and tested, with the very encouraging result that it gave 4½ million volts when operated, without the charged particle beam but with the vacuum tube in place. Work is now proceeding towards the completion of the erection and the acquirement of higher voltages.

There was taken in hand the development of certain facilities which will function with the generator and upon the ion source which will be responsible for providing the beam of charged particles.

The small 2 million volt generator continued to function very successfully and to turn out researches continuously.

In the cosmic ray field, a second trip to Churchill, Canada, secured further important information concerning the primary intensity at high altitudes, which information has given additional significant evidence concerning the magnetic field of the sun. There were also obtained data on atomic bursts initiated by the cosmic radiation at high altitudes. A number of cosmic ray investigations are under way in the laboratory.

A very significant development in the Geiger counter field has been realized and has resulted in a special appropriation from the Navy for continuation of the work.

The small linear accelerator has been brought to the stage of successful operation where it is now being used as a tool for investigations pertaining to the reflection of electrons from metals and allied phenomena. This field is also being pursued in its full generality.

Important work pertaining to thermionic emission and allied fields of interest has continued.

In addition to guiding the work of the Laboratory, the Director also functioned as Chief of the Radiological Branch of Division 7 of the Philadelphia County Civil Defense Council, with Dr. C. E. Mandeville as Deputy.

The shop and glassblowing facilities of the laboratory have continued to function effectively and to the advantage of the work of the Foundation.

The laboratory as a whole is operating under five Government contracts and its potentialities are further enhanced by its own budget.

The Biochemical Research Foundation

The year 1950 was both a trying and an exciting one at the Biochemical Research Foundation. It had looked for a return to a more nearly normal supply of mature investigators for its staff, both to replace research fellows completing their doctoral research and to strengthen our general organization in certain special fields. Unfortunately, the international situation again forced on industry and governmental agencies an extraordinary demand for scientists of all types, especially biologists and biochemists. The current growth of Northern Delaware as a vital defense production area will probably magnify this problem at the technician level. There will be continued, however, the policy of offering qualified people the opportunity to do fundamental research related to important biological problems in an environment which has been conducive to the highest type of creative investigational work, work vital to the long term security of our nation.

The new leads opened up in the Foundation's scientific work and the progress made on these developments were most encouraging. Its general emphasis on nucleic acids and the enzyme systems related to them appears to be a sound approach to the larger problems to be solved, and work is continuing along these lines. As a matter of fact, almost the entire staff is concerned directly with such nucleic acid-related problems. This focusing of different skills and types of training on one central problem has, for many years, formed the basis of this organization's research philosophy.

The number of contributions appearing from the Foundation's laboratories has continued to be high. That their significance is great is apparent from requests for reprints and from editorial comment.

The Franklin Institute Laboratories for Research and Development

The dollar volume of business from Laboratory projects in 1950 totaled $1,567,462.65. Sixty-one per cent of the proposals submitted for government work were accepted. Of the proposals submitted for industrial work, 43.7 per cent were accepted. This reflects a greater percentage of acceptance than in 1949.

Personnel showed a slight drop during the year—starting at 220 staff members and ending the year with 211—due chiefly to the increased demand for industrial manpower and for specialists in government.

There was no change in space occupied, which remained at 57,000 sq. ft., but practically every available foot of space has now been put into service.

Industrial sponsors in 1950 were: the Academy of Natural Sciences, Atlantic Refining

Company, Attapulgus Clay Company, Bethlehem Steel Company, Birdsboro Steel Foundry & Machine Company, The Budd Company, Doehler-Jarvis Corporation, The E. I. du Pont de Nemours & Company, Inc., Electric Storage Battery Company, General Electric Company, Industrial Research Laboratory, The W. K. Kellogg Foundation, Link-Belt Company, Arthur D. Little, Inc., Pennsylvania Railroad Company, Pennsylvania Salt Manufacturing Company, Redevelopment Authority of the City of Philadelphia, Philadelphia Electric Company, Pittsburgh Corning Corporation, Presto Recording Corporation, Radio Corporation of America, Raybestos-Manhattan, Inc., Superior Tube Company, Texas Company, and the H. L. Yoh Company.

New Equipment Built by the Laboratories: An advanced time-scale analog computer for analysis and synthesis of servomechanisms and communication recording equipment for study of air-ground traffic communications were built by the Laboratories in 1950. And a vibration tester was developed.

New Equipment Purchased in 1950: Major items of new equipment purchased during 1950 were for the new Hydraulics Laboratory, under which heading they appear.

Items Completed in 1950: The most important items completed during the year were graphical simulators for study of air traffic control systems and airport surface-traffic problems; a high frequency generator (financed by the Institute) to provide between 1 and 2 kilowatts of ultrasonic power for driving ultrasonic transducers, which is being used to develop industrial projects; and a torsion pendulum of unique design to extend, on a lower frequency range, studies of synthetic latex. These studies are now being made with the torsion crystal.

New Fields of Work:

Rotating Electrical Machine Design: Under a government contract, some motor designs were developed which promise to provide far better transient response than any built before.

Institute Sponsored Projects:

Gray Tin: Experiments were performed by evaporating tin on a cold surface (gray tin is stable only at low temperatures) in a vacuum unit. Preliminary identifications with X-ray diffraction show with a reasonable degree of certainty that gray tin was produced.

New Laboratories Established:

1. Magnetics Laboratory. Plans were completed for the establishment of a new Magnetics Laboratory, which will be used for basic research in the field of permanent magnetic alloys. This work will soon include magneto-strictive alloys.

2. National Asphalt Research Center. Plans formulated and executed during 1950 provided for the establishment of the National Asphalt Research Center (NARC) in January 1951, as a part of The Franklin Institute Laboratories for Research and Development. By 31 December 1950, nineteen industrial organizations throughout the United States, drawn from various fields, had joined with The Franklin Institute to sponsor the NARC. They were: Armstrong Cork Company, Atlantic Refining Company, Certain-Teed Products Corporation, Cities Service Research and Development Company, Flintkote Company, Globe Oil & Refining Company, Gulf Research and Development Company, Hood Rubber Company, The Lehon Company, The Ohio Oil Company, Owens-Corning Fiberglas Corporation, Pabco Products, Inc., The Richardson Company, Sinclair Refining Company, Standard Oil Company, of Indiana, Standard Oil Company of Ohio, Stokes Molded Products, Inc., Sun Oil Company, and the Western Molded Products, Inc. Two major projects were adopted for the first year. The first is to develop means for measuring rheological properties of asphalts over wide ranges of temperature (0°–500°) and rates of shear from creep to impact. The second is to develop a method for measuring the work of adhesion of asphalts to solid surfaces. A modern, well-equipped laboratory has been provided to carry on this work, the equipment including the new vibration tester being developed by the Laboratories.

3. Applied Hydraulics Laboratory. An applied Hydraulics Section was created and a hydraulics laboratory set up in the Laboratory Annex. The new equipment includes a *Dyna-Myke*, which is a carrier amplifying system to be used with linear differential transformers; a *Dynamometer* rated at 45 hp. as an absorption generator; a 50 kw. *Motor Generator* set; and a *High Pressure Hydraulic Unit* with a capacity of 5000 psi. at 10 gpm. Higher pressure is obtainable with an intensifier. This group is doing extensive work in hydraulic power transmission and control, and its application to industrial machinery. The group also engaged in the development and application of oil film bearings, in a study of valves and pump performance, and in special air compressor development and testing.

4. *Colloids Section.* With the opening of the NARC, it was decided to found a new Colloids Section for the study of paints, plastics, and similar materials.

Odor Abatement Project: The staff of the new Colloids Section also performed research on Odor Abatement for the Redevelopment Authority of the City of Philadelphia in minimizing the industrial odors and fumes in the Aramingo area. A method was defined for collecting and analyzing the chemicals that cause odors. This method was applied to emissions of various rendering plants, the result indicating the presence of chemicals not previously known to exist in these emissions. Recommendations to extend this work have been made by the Air Pollution Control Division and endorsed by the City Planning Commission.

Devices to Aid the Blind and the Partially-Sighted: Effort has been continued for the development of a new type of guidance device which operates on principles different from those previously applied. It is a type of cane having both curb detection and obstacle-warning features, and whose operation is based on the use of ultraviolet light. Preliminary results of laboratory tests and short term experience with blind individuals indicate that these principles when fully exploited may well provide a very useful aid to the blind in the avoidance of obstacles in walking.

Projection Magnifier: In the last Annual Report we noted the completion of a projection magnifier for the use of the partially-sighted. The instrument was sent to many schools for the blind, free libraries, and to numerous individuals throughout the United States and a few abroad. On the basis of reports received, some of which contain valuable suggestions for slight improvements, the magnifier has undergone certain modifications and has been developed to the point where it should be possible to manufacture a better model for distribution to the general public at a reasonable price. Recommendations are being made to The Kellogg Foundation, sponsors of the project, to proceed with the details of obtaining a license for their manufacture of the instrument.

Friction and Ballistics Section: Three major projects were performed in 1950 in the fields of artillery ammunition and cannon research. Work was also performed on frictional phenomena. Friction between metallic surfaces was studied at high speeds and high pressures. Numerous small projects were undertaken and completed for industry in the Physical Testing Laboratory.

High Polymer Research: Basic and applied research into the dynamic properties of polymers has been continued. These investigations are designed to correlate molecular structure and physical properties, and we have relied largely on ultrasonic techniques. In the polymer solution studies, torsional crystals have been used to study shear elasticity in the range of 20 kilocycles and with very low shear values. Particular attention has been given to the dependency of viscosity upon frequency of applied shear. Formulation of adequate mathematical models of the polymer molecules has begun. In order to extend the range of the shear elasticity studies, a new torsional pendulum has been developed to operate in the vicinity of one kilocycle. In the bulk polymer studies, using ultrasonic radiations in the megacycle range, particular attention has been given to absorption as a function of temperature and frequency. To prepare specimens for examination under the electron microscope, a technique has been developed for cutting thin sections from rubber samples. Instrumentation has been effected for measurement of physical properties.

Radiation and Thermodynamics: Instrumentation developed in connection with other infrared studies has been adapted for determining reflection spectra of materials in the infrared region. Techniques have been developed for studying the formation of shock waves as a function of time, these shock waves being formed by gases emerging from orifices at high pressures.

Considerable progress has been made toward understanding the mechanism of the combustion of hydrocarbons. In particular, a rate-controlling mechanism has been proposed to account for the oxidation and the knock-behavior of hydrocarbons in an internal combustion engine; a mechanism for investigating the action of metallic antiknocks has also been proposed. Work has been in progress on the construction of a scanning infrared spectrograph for use in the detection of certain intermediates in oxidation reactions.

Studies were continued on infrared detection, particularly with respect to the pneumatic detector; on optical and thermal properties of thin metallic films; on the thermal properties of solids, liquids, and gases; and on problems involving engineering thermodynamics and heat transfer.

Engineering Mechanics: Work was conducted in experimental and analytical stress analyses, including photoelasticity, brittle lacquers, and flow studies using Bentonite Clay solution with polarized light. Industrial investigations incorporating these techniques included structural analysis of glass bricks, pressure distribution between packing and cylinder walls, stress analysis of heavy duty machinery, and investigations of industrial processes for the purpose of improving the product and reducing costs of manufacture.

Design Section: The efforts of the large Design Section of the Mechanical Engineering Group were devoted almost entirely to the design of industrial equipment and heavy ordnance materiel.

Electronics: Work on special applications of analog computing methods was continued, especially in the field of airplane simulation for human engineering studies. Further refinements were made in a system for very precise multiplex transmission of analog data. A multichannel spectrograph was designed for airborne use, with provision for the recording of all pertinent data on conditions at the time of measurement.

Electrical Machinery: The basic problem of developing electric power drives for extremely high-performance service, such as rapid reversing mill and machinery drives, and so forth, was studied in considerable detail. Different designs of electric power transmissions were developed suitable for use as power-controlling means for high performance, automatic control equipment.

The analysis and synthesis of automatic control systems were pursued intensively. A new and different time-scale electronic analog computer was constructed and put into service. This computer permits us to set up electrical analogs of machines and controls to predetermine the best equipment for obtaining automatic performances. Applications include position and speed control systems suitable for high speed press and mill control, as well as automatic control of machine tools.

In the field of magnetic recording, studies were made of several new recording and playback heads with unusual design features. The instrumentation facilities for research in this field were expanded.

Work was done on special control devices intended to improve the security of railroad signaling equipment by improving the electrical conductivity of the juncture of the rail with the car wheel.

Navigation and Communication: During 1950, five projects dealing with air traffic control studies were worked on for the Air Navigation Development Board. In March were completed the design and fabrication of superior graphical aids for investigating the external acceptance rate of an airport; subsequently, three related projects were defined and undertaken. These comprised the fabrication of a useful graphical simulator for investigating airport surface traffic control, which was completed in September; a rigorous application of graphical aids to the analysis of realistic traffic control systems with actual daily cycles of aircraft arrivals; and the organization and conduct of two large-scale measurement programs, which are to be completed by January 1952. A communications recording truck was outfitted and tested at the Philadelphia International Airport in December, 1950, and instrumentation was assembled for a program of radar measurements of aircraft approaches.

Sonic Testing: Sonic testing was applied to an increasing variety of materials. Satisfactory attempts were made to define a mathematical expression for proper classification of grinding wheels. This expression involves various physical constants of the abrasive media and takes into account variations in the dimensions of the wheels. The applicability of a new mathematical expression for grade was tested and given tentative approval by the sponsor. Preliminary efforts to correlate the grade of grinding wheels with grinding effectiveness were not conclusive. It is expected that the problem will be solved as the work continues.

Aeronautics: An experimental model of a free-air thermometer suitable for flight test evaluation was designed, developed and constructed, using an entirely new method of free-air thermometry originated in the Laboratories.

Tours of the Laboratories: Realizing the benefit derived through closer contact with national technical societies and organizations, the Institute extended invitations to several to hold their annual meetings or periodic conferences in the Lecture Hall, and conducted their members on appropriate tours of the Laboratories. Tours were arranged for other groups during the year, such as the Pennsylvania Conference of College Physics Teachers; the winners of the local and national Science Fairs; a group from the AMC, USAFR Research and Development Detachment, Watson Laboratories, Red Bank, N. J.; supervisory employees of the Electric Storage Battery Company; and the Museum personnel of the Institute.

In an effort to assist local educational institutions, the Laboratories conducted several training tours wherein the kinds of work in which groups of students were particularly interested (physics, metallurgy, chemistry, and others) were explained. The schools represented were Drexel Institute of Technology, Temple University, St. Joseph's College, and the Engineering Institute.

Technical Papers and Publications: Technical papers were prepared by many members of the Laboratories' staff during 1950 and were accepted by authoritative publications of wide circulation, or presented before technical societies of high standing. Reprints were obtained of all of them for distribution periodically to a selected list of industrial organizations, government agencies, and technical individuals who have requested that their names be added to our regular distribution list.

Miscellaneous: A new phase of publicity for the Laboratories was the appearance on local television shows, such as *Public Invited*, of several staff members for interviews concerning different types of work being performed.

A number of staff members have been appointed to serve as officers or members of committees of several national technical societies. And the services of our staff as speakers at local clubs and technical meetings continue to be in demand.

The new *Directory of Industrial Research Laboratories of the United States* published by the National Research Council in 1950 contains a very interesting review of the work of the Laboratories. Information was also submitted to *Poor's Register of Directors and Executives*, which will be included in their next edition. Other listings, previously reported, have been maintained currently.

Securing Technical Personnel Becomes Problem: Securing qualified technical personnel became a serious problem toward the end of the year, especially in view of the many acceptances of proposals for new work which were received. A plan was started, which would be continued in 1951, whereby three staff members of high echelon would visit colleges and universities to interview students of high scholastic standing to the purpose of interesting them in joining our staff. Early reports indicate that the contacts were highly successful, not only in obtaining promising prospective employees, but in spreading the name of the Laboratories among the different educational institutions and the young scientists and engineers of tomorrow.

In conclusion, it is noted that during 1950 there was an over-all gain in activities and a gain in work load pushing our available space more severely than ever before. A sense of satisfaction and encouragement has been derived from the fact that we have been able to earn and maintain a position of prominence in basic research and development.

Finances

The financial position of The Institute at December 31, 1950 as compared with December 31, 1949 is reflected in the following comparative summarized statement adapted from the report of our auditors, Messrs. Lybrand, Ross Bros. & Montgomery, a complete copy of which is available for examination by members in the office of the Treasurer:

Assets	*Dec. 31, 1950*	*Dec. 31, 1949*
Operating Funds		
Cash	$ 64,322	$ 150,628
Undistributed income due from Endowment Funds	43,197	31,223
Amounts due on Laboratory research contracts	438,941	338,979
Other receivables	31,560	24,463
Inventories	33,908	30,946
Deferred charges	21,143	12,534
Total operating funds	633,071	588,773
Endowment Funds		
Investments, at book value (market values $4,635,096 and $4,071,319 respectively)	3,625,531	3,477,089
Uninvested principal cash	25,479	33,187
Due from Operating Funds	75,000	75,000
Due from Plant and Property Funds	220,000	295,000
Income cash	101,977	119,933
Total endowment funds	4,047,987	4,000,209
Plant and Property Funds		
Cash	784	855
Building, equipment, fixtures, etc. (less reserves for depreciation of $669,661 and $490,272 respectively)	3,844,011	3,993,946
Due from Operating Funds	316,531	241,525
Total plant and property funds	4,161,326	4,236,326
Total assets	$8,842,384	$8,825,308

Liabilities and Funds		
Operating Funds		
Accounts payable, accrued expenses, deferred income, and reserves	$ 178,113	$ 161,217
Due to Endowment Funds	75,000	75,000
Due to Plant and Property Funds	316,531	241,525
Capital stock	10,580	10,690
Operating Fund balance	52,847	100,341
Total operating funds	633,071	588,773
Endowment Funds		
General and restricted endowment funds and funds functioning as endowment including those of Bartol Research Foundation	$3,946,010	$3,880,276
Undistributed and unexpended income	101,977	119,933
Total endowment funds	4,047,987	4,000,209
Plant and Property Funds		
Contributed funds	3,941,326	3,941,326
Due to Endowment Funds	220,000	295,000
Total plant and property funds	4,161,326	4,236,326
Total liabilities and funds	$8,842,384	$8,825,308

The results of general operations for the year 1950 compared with 1949 are shown in the following comparative statement of Income and Expenses of Current Funds:

	1950	*1949*
Operating Revenue:		
Membership dues	$ 56,130	$ 56,577
Museum admissions and other Museum income	101,016	95,194
Laboratory research projects	1,567,463	1,743,714
Revenue from other activities	45,068	51,580
	1,769,677	1,947,065
Operating Costs and Expenses:		
Museum	265,036	271,590
Library, less amounts transferred to other depts.	61,483	54,566
Laboratory research projects	1,492,975	1,702,415
Other activities	69,758	82,448
Administrative and general expenses, less amounts transferred to other departments	62,695	59,349
	1,951,947	2,170,368
Operating loss	182,270	223,303
Other Income:		
Income from endowment funds	93,597	77,291
Gifts from Friends of Franklin	25,443	22,120
Appropriation, Commonwealth of Pennsylvania	38,500	36,833
Other income	16,903	15,934
	174,443	152,178
Deficit before other charges	7,827	71,125
Other Charges:		
Interest	9,388	12,403
Transfer to Endowment Funds	16,100	2,030
Other charges	14,290	5,657
	39,778	20,090
Excess of costs and expenses over income transferred to Operating Fund balance	$ 47,605	$ 91,215

The excess of costs and expenses over income shown above is after charges for depreciation or amortization of buildings and equipment amounting to $180,843 in 1950 and $178,127 in 1949.

During the year a bequest of $5000 under the will of the late Harry L. Jefferys was received with gratitude. This bequest was for the purpose of maintaining and utilizing the philatelic collection which he left to The Institute.

As has been pointed out in recent reports The Institute needs funds for the development and enlargement of its Museum educational activities; for engaging in self-sponsored research work in its Laboratories; for the addition of books for its Library—and particularly for the binding and cataloguing of Library material to make it of real use to the public; and of course for carrying on its general routine, undramatic operations. The Institute appeals to those who believe in its work to help toward its support.

Respectfully submitted,
By order of the Board of Managers,
RICHARD T. NALLE,
President

LIBRARY

The Committee on Library desires to add to the collections any technical works that members would wish to contribute. Contributions will be gratefully acknowledge and placed in the library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat Service. Photostat prints of any material in the collections can be supplied on request.

The Library and reading room are open 9 A.M. until 5 P.M. on Mondays, Tuesdays, Wednesdays and Fridays; 2 P.M. until 10 P.M. on Thursdays; and 9 A.M. until 12 noon on Saturdays.

RECENT ADDITIONS

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THE FRANKLIN INSTITUTE LABORATORIES FOR RESEARCH AND DEVELOPMENT

A TEN CHANNEL INFRARED SPECTROGRAPH

During the past ten years, especially since the end of World War II, there has been much activity in the field of development of rapid infrared scanning spectrographs. This activity has been due to the need for spectral distribution data, in the infrared region, as a function of time. For example, data of this type would provide information concerning the reaction mechanism of rapid combustion processes.

The development, by Dr. Marcel Golay and a staff of Signal Corps personnel, of the pneumatic type infrared detector represented a large step forward in the field, since it resulted in a very versatile and fast response-time detector, with grey response in the range 1–20 microns.

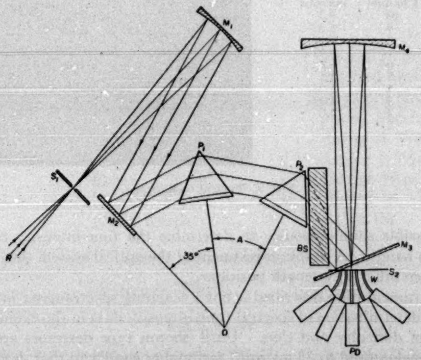

FIG. 1. Optical system of infrared spectrograph (plan view). M_1, M_4, parabolic mirrors; M_2, M_3, plane mirrors; BS, beam splitting mirrors; P_1, P_2, NaCl prisms; S_1, entrance slit; S_2, exit slits; W, wave guides; PD, photothermal detectors; A, adjustable angle; O, pivot point for Wadsworth mounting and second prism; R, incoming radiation.

***Abstract of* A Ten Channel Infrared Spectrograph.**[1]—JOHN T. AGNEW, RACHEL G. FRANKLIN, AND ROBERT E. BENN. This paper describes one approach to the problem of the development of a fast infrared spectrograph. (See Fig. 1.) Ten of the pneumatic type detectors are mounted on a bronze hemispherical support in such a way that their optical axes intersect at a common point approximately 4 in. beyond each detector. Energy, dispersed by two sodium-chloride prisms arranged in a double Wadsworth mounting, is focused in a plane which contains ten exit slits formed by twenty tiny slit jaws in two rows. (Fig. 2.)

FIG. 2. View of spectrograph showing prism tables and exit slits.

The energy passing through each exit slit is "piped" to a corresponding detector by means of internally polished silver tubes.

The electronic signals generated by the pneumatic detectors are amplified and fed through a ten-channel electronic switch which simultaneously presents all ten signals on a single oscilloscope screen. The time sharing feature of this type of presentation provides an automatic timing feature. (Fig. 3.)

Channel	Wavelength microns
10	9.3
9	8.32
8	7.16
7	6.62
6	5.8
5	5.1
4	3.94
3	3.22
2	2.0
1	1.1

FIG. 3. Typical oscilloscope.

Thus it is possible simultaneously to determine the time-intensity relationship for ten narrow wavelength bands. A simple adjustment of the optical system then allows an identical determination at ten other wavelength positions.

While the instrumentation described is not a scanning spectrometer in a true sense of the word, it allows the determination of spectral-time-intensity data in the extended infrared region with a minimum of difficulty and time. Until photon type detectors are developed in the extended infrared region (and to 25 microns) techniques similar to that described in this paper will be necessary to obtain the information needed.

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[1] *Journal of the Optical Society of America*, Vol. 41, No. 2, pp. 76–79, February, 1951.

V. 19

JOURNAL OF THE FRANKLIN INSTITUTE

The following papers will appear in the JOURNAL within the next few months:

HIGGINS, THOMAS JAMES: Solution of Boundary Value Problems by Multiple Laplace Transformations.

HUNICKE, A. BYRON: The Valved-Tremie Applied to Subaqueous Concrete Structures.

RAMSEY, W. E.: The Slow Discharge in a Non-Self-Quenching Geiger-Mueller Counter.

ZETROUER, W. F. AND WILLIAM J. KESSLER: A Note on the Similarity of Certain Atmospheric Waveforms.

PATAI, IMRE AND MARTIN A. POMERANTZ: Contact Potential Differences.

KRZYWOBLOCKI, M. Z.: On the Foundations of Certain Theories of Turbulence.

SPENCER, DOMINA EBERLE AND VINCENT J. STAKUTIS: The Integral-Equation Solution of the Daylighting Problem.

SILBERSTEIN, LUDWIK: Superposition or Mixture of Simple Quantic Characteristic Curves of Photographic Emulsion Layers.

KNOWLES, T. AND A. H. TAYLOR: Spectral Radiation Involved in Photoreactivation of Ultraviolet-irradiated Cultures of Micro-organisms.

SYMONDS, P. S.: Recent Progress in the Plastic Methods of Structural Analysis.

MEMBERSHIP

ACTIVE MEMBERS ELECTED AT THE MEETING OF THE BOARD OF MANAGERS

June 20, 1951

ACTIVE LIFE NON-RESIDENT

Frederick Fraley, Jr.

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Rudolph L. Hellmund
N. L. A. Martucci
Richard C. Meyer
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James B. Williams
Lealor J. Wilson
George Wustner
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Warren M. Hagist

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J. Alexander Kline '42
Joseph W. McGinley '36
Reginald Norman '99
Mrs. Charles J. Rhoads '35
Cornelius J. Ridgley '47
J. Leonard Sessler '43
R. E. Taggart '47
Clarence A. Warden '37

MUSEUM

MECHANICAL SOLAR SYSTEMS

Vast distances and huge masses have always fascinated the mind of man and, after Copernicus had outlined his theory of the solar system, men have striven to show by mechanical means how that system worked. The earliest representation of the solar system in the possession of the Museum is that shown on the Strasburg clock. The original clock, completed in 1354, had a dial which displayed the movements of the Sun and Moon around a stationary Earth. When the clock was rebuilt in 1571–1598 it bore a portrait of Copernicus and proved that his system of the universe had found favor with the makers when a crude orrery was included among its features. When the clock was again rebuilt in 1842, the orrery was included, and may be seen in the model of the famous clock which is exhibited in the Hall of Astronomy.

Sometime around the year 1680, Olaus Roemer, a Dane, created a mechanical planetarium which provoked the great Dutch scientist Christian Huygens to think of improving upon the example. Huygens invented the mathematical method known as "continued fractions" in order to determine the proper number of teeth for the gears of his planetarium, thereby paving the way for a more accurate representation of the motions of the planets. Inspired by this achievement the Englishman John Rowley, in 1715, constructed for Charles Boyle, Earl of Orrery in Ireland, an improved instrument. Because of association with the owner, these instruments have since been known as "orreries."

Thereafter, the production of new and improved orreries accelerated and in 1732 what is thought to be the first of its kind in America was brought here. Unfortunately, the maker of this interesting relic of bygone days remains unknown, although it had several distinguished owners. Legend says that it was actually constructed by Thomas Jefferson, in whose possession it remained for some time. This delicate piece of mechanism is also displayed in the Hall of Astronomy and deserves more attention than it receives as an historic piece of astronomical apparatus.

However, the most impressive orrery on display in the Museum is the beautiful Sendtner instrument which stands in the Planetarium Lobby. Enclosed in a star-studded five foot globe the Sendtner orrery is a masterpiece of the mechanic's craft, with its complicated gearing and skillfully turned rods. All the planets are represented except Pluto, which had not been discovered at the time the orrery was constructed. However, the visitor who may marvel over the beautiful workmanship can be re-assured that the representative speeds and orbital periods of the various planets have been worked out and maintained with an uncanny precision.

Viewing the solar system from any one member of the satellites does not give the observer a true picture. In order to appreciate the relative distances and speeds of motion, the observer has to be transported to a point outside the entire system. This is the purpose of the orrery. It enables an observer to stand in imagination outside the solar system and look inward upon its various members. By the exercise of a little imagination he can magnify the distances and the relative masses of the planets and acquire a more or less accurate picture of what takes place within that corner of the universe which we call our own—the solar system. Familiarity with an orrery is an excellent introduction to the more elaborate presentations of the Fels Planetarium, where the lecture-demonstrations help to explain the details which cannot be learned from a study of the orrery, the Planetarium instrument being of infinitely greater flexibility, size, and power.

V.
19

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION

ELLICE McDONALD, DIRECTOR

The following is a leading article editorial in the November 18, 1950 number of the great English scientific journal *Nature*. There have been hundreds of requests from 17 different countries for "Research and Its Organization."

LEADERSHIP IN SCIENTIFIC RESEARCH

The series of papers on research and its organisation by Dr. Ellice McDonald, director of the Biochemical Research Foundation of the Franklin Institute, originally published in the *Journal* of the Institute, have recently been re-issued under that general title, together with an earlier paper on a bonus system in a research organisation, and two more recent papers on biochemical and medical research, one by Dr. McDonald and one by Dr. W. A. Mosher*. Covering the choice of research projects, co-operation in research, the direction of research and the choice of workers in research, these papers make a contribution to the literature on the administration and organisation of research, which is the more useful for being drawn from the author's personal experience and quite independent of other current writing in this field.

In this brochure Dr. McDonald discusses first the changing face of research and, after commenting that the influence of the research worker on the community is indirect rather than direct, emphasizes that the changes in recent years are due first, to the annihilation of distance, then to the increased scale of operations and, third, to the increased use of power. The improvement in rapidity and ease of communication due to the first factor has accelerated the progress of research to an extraordinary degree, while the increased scale of operations has not only encouraged the extension of team-work and of large co-ordinated groups, but has also had its effect on the exchange of information. The increased use of power has had its effect chiefly in the large numbers of instruments of precision which have been devised to extend the research worker's range of observation.

Dr. McDonald argues that the influence of these changes has been most marked in the field of development rather than in that of discovery and research, and he questions whether a planned and completely co-ordinated research programme on a national scale would offer any advantage over independent research activity. Discoveries are made by individuals—men and women—and if the free spirit of research is hampered, vision is impaired. A planned research tends to concentration in much detail in a few directions, and accordingly, he argues, the chance of hitting the correct solution is less.

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*"Research and its Organization," by Dr. Ellice McDonald. Pp. 96. (Newark, Del., Biochemical Research Foundation, 1950.)

The chief stress, however, is laid by Dr. McDonald on the creative aspect of research and the need for a sympathetic understanding of the research instinct and for untrammelled development of its creative genius. This, he urges, is a main factor to be taken into account in choosing research projects. The proper choice of research topics, he rightly insists, is the vital factor determining success in research; he defines research projects as experiments directed by the disciplined imagination of a group of individuals of varied mental outlook and diverse training, who are able to transcend the results obtained by the isolated scientific worker, save for the exceptional genius. Commonplace qualities, well trained, well proportioned and consistently well displayed, will in organised groups, he maintains, give consistently better success than any individual effort.

The dangers and limitations in the planning and organisation of research are clearly well understood by Dr. McDonald; in this connexion his thought seems to run on similar lines to that of Dr. E. F. Caldin, in his recent book, "The Power and Limits of Science." He writes wisely on the postulates to be considered in selecting a research project —and no less in abandoning it—and here again the effect on the staff concerned is uppermost in his mind. While recognizing that many important discoveries have been made by accident, he points out that by far the greater number have been made by design, and to his mind the vital task appears to be so to present accumulated and recorded experience to the next generation of research workers that they live in an atmosphere of intellectual stimulation and are inspired to imaginative activity.

V. 19

The very wealth of that experience, however, enforces the need for co-operation in research as in other fields. Coupled with the vast increase in the complexity, range and expensiveness of the tools available for research, it has made organisation imperative if the size of operations is to match the new opportunities. Dr. McDonald recognizes that we are only now discovering the art of working together, and he defines the art of good management as the creation of an organisation which will develop to its highest point the initiative of individuals, consistent with an orderly progression and cohesiveness of purpose. The purpose of management and of the collective intelligence, experience and imagination (which imply direction to a given scientific task in research) is likewise to expand the power of each individual worker to be useful; and in this connexion he has in mind particularly the facilitation of co-operation and the interchange of experience between different specialists and across the frontiers of different disciplines. Many gaps in the uneven front of research to-day, he reminds us, are represented by unexplored fields which lie between the activities of two or more branches of science. These gaps can only be filled in by the interchange of ideas from day to day and over a long period of time, because it is difficult for men of science in different fields to learn the *patois* which each speaks. A large research organisation is one means by which that difficulty can be overcome, and when due regard is had to the inherent weaknesses of such an organisation to which Dr. McDonald directs attention, the risk of it cramping originality of thought or repressing the flair for discovery is small.

Under present conditions, a certain amount of organisation in research is inevitable. The essential task is how to minimize its weaknesses and drawbacks; and much in the first place depends on the manner in which the direction of research is exercised, and especially on the director of research himself. Dr. McDonald's ideas on this point are clear and incisive. He has no delusions or exaggerated ideas on the part the director or research manager has to play. It is an administrative task calling for someone who, while sufficient of a man of science to understand the outlook of the research worker, is prepared to set on one side any personal interest in research and to take upon his shoulders the bother of the details inherent in an organisation, so that the research workers themselves may be relieved and free to concentrate their energy on the prosecution of their investigations.

Such a task calls for all the qualities of leadership in the full sense of its meaning—ability to understand and to handle men, sensitiveness to the atmosphere in which research is most effective and creative, and unflagging zeal to secure and maintain the appropriate atmosphere as well as the mechanical and intellectual tools required. A director of research to-day is required, too, to take account of the outlook and thought of scientific workers of diverse character and training if he is to weld them into a successful team. The planning of resources in men and women, as well as in material equipment, must figure in successful strategy.

Much here will, as Dr. McDonald points out, depend on the wise and careful choice of workers, though besides choosing the best it may also be necessary to get rid of those who fall short of the standard. No list of qualities—judgment, common sense, modesty, originality, precision, and an unquenchable curiosity are all indicated by Dr. McDonald—can replace the capacity to estimate men and their ability, though experience has a valuable part to play. In the paper in which he discusses the selection of research workers and the qualities they should possess, Dr. McDonald is not completely happy, and in one passage he lapses into a gibe at the amateur. He admits the difficulty of recognizing genius; but in claiming that the lone research worker has a span of only three years in which his ideas are prolific, he seems unduly pessimistic. He suggests that in an appropriate organisation the stimulating contact with other men of science and other fields of work can prolong the creative life of the individual. If, as Prof. C. J. Renier wrote in his recent book, "imagination is nothing but the mobilization of the contents of our memory," there may be something in this view. Much will still depend, however, on personal qualities or characteristics if the effective group co-operation and stimulus are to be secured. With the exceptional worker—and it is the man or woman with outstanding ability in research that Dr. McDonald appears to be considering—this cannot be taken for granted. He recognizes indeed that some seem to find impossible the give and take which is required in group research, and there are others who will require educating in this respect before they learn that they, as well as the group, profit from the interchange of ideas and from contact with other branches of science. Some few, indeed, will have no place in the modern scheme of research.

In these highly suggestive papers, Dr. McDonald has both emphasized some of the most important factors upon which success in research depends and has also indicated some of the directions in which still further knowledge is required. We are still far from understanding fully the conditions for the stimulation of creative work in scientific research, particularly that concerned with several disciplines or branches of science.

BOOK REVIEWS

FERROMAGNETISM, by Richard M. Bozorth. 968 pages, illustrations, 16 × 24 cm. New York, D. Van Nostrand Co., 1951. Price, $17.50.

This book is another in the excellent Bell Telephone Laboratory series. It represents a really tremendous contribution by Dr. Bozorth and the Bell Laboratories' staff to progress in ferromagnetic studies, because within its covers are brought together in great detail information on the metallurgical and magnetic properties of ferromagnetic materials, and a generous survey of present knowledge of the nature, both empirical and theoretical, of the phenomena of magnetism. It is a book, as the author states in the preface that he wishes it to be, that will be useful to the physicist interested in the fundamentals of ferromagnetism, to the metallurgist working with magnetic materials, and to the design engineer adapting new methods and materials to apparatus needs. The book is characterized by completeness in coverage, rather than by depth. This is especially noticeable in the sections dealing with theory, where generally the results only of the essentially mathematical theory are given. Nonetheless, even here it is extremely valuable as a survey of the field and as a source for further references.

The first half of the book is given over to descriptions of ferromagnetic materials, which are again subdivided into non-permanent and permanent magnetic materials. A general section describes production and metallurgical processes. Then various pure materials and alloys are considered one by one. Physical and magnetic properties are described and tabulated for each; whenever possible, these properties are related to the phase diagram of the alloys. The effects of impurities and of heat treatments are considered. Methods of purification and of analysis are given. In the permanent magnet materials section, some attention is given to design of permanent magnets and the type of engineering data used. In addition to permanent magnet alloys, fine powders and ferrites are also discussed.

The second half of the book deals mainly with a description of magnetic phenomena and the underlying theory. Ferro-, para-, and dia-magnetism are discussed in general. The details of ferromagnetic behavior as explained by domain theory occupy a chapter, as do the magnetic properties of crystals. Several are devoted to magnetostriction. The thermodynamics of magnetization, including the behavior of phase changes at the Curie temperature, and the general energetics involved in magnetism, are discussed. Time dependence of magnetic effects, including long-time effects such as aging and magnetic viscosity, and short-time effects such as ferromagnetic resonance, are discussed. This section is concluded with consideration of some special problems in domain theory.

A short chapter on experimental techniques and apparatus used in making magnetic measurements closes out the text. Four appendices give physical properties of the elements, values for some constants, and magnetic properties of some special magnetic materials, together with their composition and manufacturer. An extremely impressive bibliography is given. It includes over 1700 references, dating back as far as 1842 and extending up to 1951.

The outstanding characteristic of this book is its usefulness. It is a tremendous source of data, a compilation that is very well organized. The data are presented in the form of illustrations of properties of materials, or of physical principles or theories. They are given mostly in the form of diagrams and graphs. The author claims almost 0.9 graph per page; paging through the book, one wonders where the other 0.1 graph went, for every page seems to have at least one. The book is, in summary, an exceptionally well documented survey of all of our present knowledge of the field of ferromagnetism. This reviewer recommends it highly to the worker in this field—it is in the don't-be-without-it class. For those in other fields, it is perhaps too detailed to be recommended as review studying.

A. D. FRANKLIN

THE FRICTION AND LUBRICATION OF SOLIDS, by F. P. Bowden and D. Tabor. 337 pages, 16 × 24 cm. London, Oxford University Press, 1950. Price, $6.30.

For more than a hundred and fifty years the theory that sliding friction is caused by the intermeshing of surface roughnesses has held sway; but doubts arose when this theory proved inadequate to explain some of the results of modern research. The theory of molecular attraction, suggested but discarded by Coulomb, was advanced by several investigators but needed much experimental proof before it could receive general acceptance. This proof has been accumulating during the past fifteen years as Bowden and Tabor, and their co-workers, have published one important paper after another. The results of these researches have now been set forth in a monograph which includes also a discussion of the behavior of lubricants as applied to shop practices and running machines.

The authors have done much to establish the theory that when one metallic surface slides over another the two are in actual contact over a very small area involving only the tops of their roughnesses which are deformed plastically by the load. These points of contact become welded together often at a low temperature but sometimes at very high local temperatures. When sliding occurs the welds are sheared off and thus bring about frictional resistance. Any film between the surfaces, such as adsorbed gases, oxides, a soft metal, or oil, acts as a lubricant because it separates the surfaces; hence fewer welds are formed.

In boundary lubrication peaks of surface roughnesses protrude through the layer of lubricant and produce welds, but the bulk properties of the metal, the polarity of the lubricant molecule and its chain length, together with the chemical action of the lubricant, all introduce variable factors.

It appears that the lubrication of metals by fatty acids is most effective only when the acids react with the metal to form the soap. These acids have a polar group that attaches itself to the metal leaving the non-polar tails to form a monomolecular layer held together by lateral adhesion. Such a "solid" surface film is a good boundary lubricant until the temperature is raised so that it "melts" and immediately loses its lubricating properties.

The authors have been most successful in compressing a vast amount of well-illustrated material into approximately 300 pages with a minimum of mathematics and without sacrificing an easy, readable style. This mass of evidence supports in a convincing manner their theory that the fundamental phenomenon of friction is molecular attraction which causes local welds that must be sheared off by application of a frictional force.

FREDERIC PALMER, JR.

LIQUID EXTRACTION, by Robert E. Treybal. 422 pages, illustrations, and diagrams, 16 × 24 cm. New York, McGraw-Hill Book Co., Inc., 1951. Price, $7.50.

This book is another of the Chemical Engineering Series which covers completely the subject of liquid extraction. The author has compiled all the important information on the subject in an organized form.

The introductory chapter describes briefly methods of separating components of a liquid solution and compares the advantages of these different methods. The chapters that follow give the potentialities and limitations of liquid extraction and can be divided into three parts. The first part pertaining to theory, deals with the physical chemistry of liquid-liquid equilibria, prediction of equilibria, thermodynamics of nonideal solutions, choice of solvent, diffusion in liquids, and theoretical discussion of mass transfer.

Methods of calculation and design of extraction systems comprise the second part of the text and cover the different types of solvent systems. Included are single solvent systems which consist of, or which may be reduced to, the equivalent of three components. Those arrangements employing a solvent solution consisting of at least two components, where simplification to the equivalent of a ternary system is not feasible are treated under mixed solvents. Finally, double solvents are considered as those arrangements where the mixture to be separated is distributed between two immiscible solvents and contain at least four components.

The topics treated in the remainder of the book are description of equipment and operating characteristics and a review of important modern chemical processes using liquid extraction. The various types of equipment are classified into those which provide discrete stages, where the liquids are mixed, extraction allowed to occur, and the insoluble phases are settled and separately removed, and into those which provide continuous countercurrent contact between the insoluble phases. Typical industrial applications are considered briefly to show the special usefulness of extraction as a separation technique. No attempt is made to treat all possible liquid-extraction processes.

The book is designed to be used as a text for study and teaching purposes as well as a reference book. Numerous illustrative examples are worked out in detail and problems are provided for practice work. Many diagrammatic illustrations, detailed tables of notation and literature references are given for each chapter.

MICHAEL MATTIA

ELECTRONS AND HOLES IN SEMICONDUCTORS, by William Shockley. 558 pages, illustrations, 16 × 24 cm. New York, D. Van Nostrand Co., Inc., 1950. Price, $9.75.

For a number of years solid state physicists have used the concepts of the excess electron and the positive hole, which is the vacancy produced when an electron is removed from the valence bond structure of a crystal, to explain the mechanism of electrical conduction in semiconductors. However, it was not until the appearance of the transistor in 1948 that the great practical significance of these principles was fully recognized. Since the transistor has many uses, both actual and potential, interest in the branch of solid state physics which Dr. Shockley terms "transistor electronics" has increased greatly.

This book has been written in such a way that it will be of value to a wide variety of persons, from the engineer interested in the practical applications of the transistor to the mathematical physicist seeking an explanation of the properties of semiconductors in terms of the quantum theory. To achieve this purpose the author has divided the book into three parts.

Part I introduces the reader to the subject of transistor electronics. Both the filamentary and the type-A transistors are described. Emphasis is placed on experimental results. The theoretical discussion in this section is elementary in character, considerable use being made of analogies to aid the reader in understanding some of the more abstract concepts. A number of topics applicable to transistors such as noise, equivalent circuits and static characteristics are discussed. These topics will be of particular interest to electrical engineers interested in the practical aspects of the subject.

Chapters 5 through 12 constitute the second part of the book. This section of the book is devoted to the theory of semiconductors, treated chiefly from the descriptive point of view. Some of the topics discussed in this section are: energy bands, the behavior of electrons and holes in electric and magnetic fields, the theory of conductivity and the Hall effect, the application of the Fermi-Dirac statistics to semiconductors and the application of semiconductor theory to transistor electronics.

The remaining five chapters comprise Part III. Here the author shows how quantum theory leads to the abstract concepts of holes and electrons. The first two chapters of Part III are intended as an introduction to the remaining discussion. It is the purpose of these chapters to bring the mathematical treatment which follows within the grasp of persons who do not have an extensive background in mathematical physics. It is the results of these mathematical investigations which are discussed in a descriptive manner in Part II.

The book is well written and easy to read. It contains numerous graphs and pictorial diagrams which greatly aid one in visualizing the processes involved. The numerous problems included in the book serve to amplify and extend the material of the text. Dr. Shockley has done an excellent job in covering this important phase of solid state physics. The book is to be highly recommended in every respect.

M. D. EARLE

THE PHYSICS OF POWDER METALLURGY, edited by Walter E. Kingston. A symposium held at Bayside, Long Island, August 24–26, 1949. 404 pages, illustrations, 16 × 24 cm. New York, McGraw-Hill Book Co., Inc., 1951. Price, $8.50.

A number of books have been published recently which are in the nature of a symposium—a series of papers by outstanding authorities in a particular field. This method of writing a book has the advantage of obtaining the best possible coverage of a subject. Frequently, of course, there is some overlapping of material, but generally it aids in more thorough comprehension. *The Physics of Powder Metallurgy* has been prepared in the method mentioned and contains papers by internationally known personalities in the field of powder metallurgy.

By bringing together metal powders under suitable temperatures and pressures, solid objects can be produced. The study of this phenomenon is known as powder metallurgy. It is a comparatively new science, and much remains to be learned before the phenomenon is completely understood. *The Physics of Powder Metallurgy* contains papers describing the up-to-date state of the science, or perhaps more accurately, the industry.

The papers range from such general and fundamental subjects as sintering, crystal growth and recrystallization to such particular subjects as shrinkage of synthetic pores in copper and sintering of copper-gold alloys. In between, papers deal with grain boundaries, self diffusion in iron, homogeneous nucleation, sintered material made up of a single atom type, diffusion and viscous flow in metals, sintering with a liquid phase, surface tension, hot pressing, sintering of carbides, and electrical properties as indicators of the degree of sintering. Generally, each paper is founded upon experiment, and consequently many fundamental, empirical data are presented. Many more data are referenced at the end of each chapter so that the book serves as an excellent reference for theoretical and experimental material on powder metallurgy. Also, following each paper are discussions of the included subject, so that a fairly broad perspective is available.

E. W. HAMMER, JR.

SERVOMECHANISMS AND REGULATING SYSTEM DESIGN, by Harold Chestnut and Robert Mayer. 505 pages, illustrations, 15 × 24 cm. New York, John Wiley & Sons, Inc., 1951. Price, $7.75.

The military needs for automatic control have provided considerable impetus to the development of a wide variety of control systems. Feedback control systems have become an integral part of modern industrial process control, and their widespread use makes the automatic factory a thing of the present rather than the future. This book is a welcome addition to the literature of this field, and is intended for the training of design and application engineers in the principles of feedback control. Furthermore, the book is adapted to the needs of those engineers and engineering students who have not had previous training or experience in the field of closed loop control systems. The authors develop the mathematics required to evaluate both the steady state and transient response of systems, and then proceed to derive the transfer functions for the various components of electric, hydraulic, and mechanical systems. Finally techniques for the synthesis of practical systems are discussed and illustrated. Numerous problems serve to illustrate and extend the text material. In addition the solution of these problems will yield solutions to many of the practical problems confronting the designer of feedback control systems.

In addition to fulfilling the aim of the book, as described above, the authors have included material that has not been previously published in book form. This material, which is of considerable practical utility, is contained in the last chapter of the book, and has the same title as the authors' AIEE Industry Class prize winning paper of 1949, "Comparison of Steady State and Transient Performance of Servomechanisms."

M. R. AARON

THE CHEMISTRY OF HYDRAZINE, by L. F. Audrieth and B. A. Ogg. 244 pages, 14 × 22 cm. New York, John Wiley & Sons, Inc., 1951. Price, $5.00.

Hydrazine, one of the simplest hydronitrogens, and its derivatives have long been considered as rather specialized laboratory chemicals. Little use other than as a research tool had been made of these very interesting compounds. During World War II the Germans found that hydrazine was useful in making rocket fuels. For this reason intensive research and development of efficient methods for synthesis of hydrazine, and hence studies of its physical and chemical properties, have been carried out since the end of the war. This book, actually a revision of an Office of Naval Research report, presents the most recent advances in the chemistry of hydrazine.

The text can be divided into five main parts: (1) the preparation and physical-chemical properties of aqueous and anhydrous hydrazine, (2) hydrazine as a reducing and oxidizing agent, (3) the qualitative and quantitative determination of hydrazine, (4) inorganic compounds of hydrazine, and (5) potential uses for hydrazine and its derivatives.

Throughout the book the authors base the organization of material and development of subject matter on the fact that hydrazine is a compound with a nitrogen to nitrogen bond and belongs in a borderline field of both organic and inorganic chemistry. Bibliographies are presented at the end of each chapter for readers interested in obtaining more detailed information on specific subjects. The book is carefully prepared and shows the great interest the authors have in nitrogen chemistry. It is highly recommended as a text-book and a source of valuable information for research workers in this field of chemistry.

D. H. RUSSELL

INTRODUCING THE UNIVERSE, by James C. Hickey. 154 pages, plates, 14 × 21 cm. New York, Dodd, Mead & Co., 1951. Price, $3.50.

Professional astronomers as a general rule turn out to be rather poor writers and as a result the elementary or beginner's books on astronomy they write are elementary only to those who know the subject, and consequently have no need for the books.

In *Introducing the Universe* James C. Hickey, who for a great many years wrote an astronomy column in the *N. Y. Sun*, has written an elementary astronomy which has been pointed for the beginner. Astronomers in general and planetarium lecturers in particular are often asked to recommend a general astronomy for the layman. This book is the answer.

Here, in a rather small book, are collected most of the answers the layman asks of the professional. Here are given the details of the planets, our moon, eclipses, meteors, asteroids, the Milky Way, and finally there is a very brief description of what lies out beyond the Milky Way. The writing is fast, enthusiastic and accurate and as such makes a readable book about the most fascinating of all the sciences.

I. M. LEVITT

THE ATOM AT WORK, by Jacob Sacks. 327 pages, illustrations, 16 × 24 cm. New York, The Ronald Press Co., 1951. Price, $4.00.

Dr. Sacks is eminently successful in accomplishing the intended purpose of the book herein reviewed, namely, to present a well-rounded and rather complete story of atomic energy in a readable and understanding manner, in layman's language. Nor does this detract from its interest to a scientist or engineer. The story of atomic energy from the early concepts of the structure of the atom to its present day uses is presented in such a manner that would allow a non-scientist reader to speak intelligently about many facets of a complex subject.

As the story of natural radioactivity unfolds, the reader is made acquainted with early experiments which led to the development of the particle theory. These are followed by a discussion of Einstein's equation relating mass and energy. The special theory of relativity is introduced and related to experiments which indicate the importance of this theory to particle physics.

Sufficient background material is presented to cause the reader to inquire as to more modern methods of production of radioactive matter. The "machines for modern alchemy" are described, from the early ones to the late, including electrostatic generators, linear accelerators, cyclotrons, synchrocyclotrons and the synchrotron. To a layman and no doubt to technical readers who may not be directly involved in atomic energy work these devices of breath-taking capabilities in the physical world are truly wondrous. From them stream a galaxy of high energy particles which are put to numerous uses, most of them beneficial to mankind.

Neutron production naturally leads to products of artificial radioactivity and their application in many fields of science. Singled out for detailed discussion are examples drawn from the chemical and biological laboratories, hospital and clinic laboratories which study many human diseases, industrial measurements, and atomic warfare. It is quite pleasing to read a book dealing with atomic energy without acquiring the feeling that here is a potential destroyer of human life.

To one who desires to obtain a reasonably good perspective of the role of atomic energy, accompanied by historical background material, this book is wholeheartedly recommended.

S. CHARP

Life Insurance Mathematics, by Robert E. Larson and Erwin A. Gaumnitz. 184 pages, 15 × 24 cm. New York, John Wiley & Sons, Inc., 1951. Price, $3.75.

There exists a need for texts as well as for mathematicians in the actuarial field. This book is intended as a textbook in life insurance mathematics, and pre-supposes no mathematical preparation beyond the high-school level though a good background at that level is required.

It is not aimed at any one of the eight actuarial examinations, and its material would not be covered before the fourth of those examinations. It is interesting to note that the second and third examinations do require college mathematics and, in general, the examinations must be taken in order. Hence, the actuarial student doesn't usually encounter this material until his second or third year in college.

The text uses new approved actuarial notation; its examples are based on up-to-date mortality tables; and a new treatment is given of the subjects of valuation and nonforfeiture benefits based on recently adopted State laws.

The book starts with a discussion of the Mortality Table. The second chapter on Interest and Annuities Certain is followed by ones on Life Annuities, Life Insurance, and Net Level Reserves. The chapter on Advanced Topics treats annuities certain payable fractionally, life annuities payable fractionally, increasing life annuities, and increasing insurance. The methods of modified reserves treated include the full preliminary term, commissioners reserve valuation, the Ohio, Canadian, Illinois, New Jersey and the Select and Ultimate Method. The concluding chapters are on Surrender Values and Gross Premiums. Three appendices complete the book: Net Level Terminal Reserves, Comparison of Natural and Net Level Premiums, and Tables.

Donald B. Houghton

Fluid Mechanics, by Victor L. Streeter. 366 pages, diagrams, 16 × 24 cm. New York, McGraw-Hill Book Co., Inc., 1951. Price, $5.00.

All the usual material in a first year college course in fluid flow is contained in *Fluid Mechanics*. The opening chapters concern pressure intensity, static pressure forces and the basic fluid flow concepts. Succeeding chapters deal with fluid jets, laminar flow, flow measurement, steady flow, flow in open channels, flow around immersed bodies and oil hydraulic systems. This last describes the fundamental concepts of hydraulic pumps, motors and systems, enumerating their advantages over other systems, and their uses.

Professor Streeter, Director of Fundamental Fluid Research, Illinois Institute of Technology, is a recognized authority in fluid flow. His material, presented in an unusually clear and sound manner, and explained by excellent and frequent diagrams and pictures and many illustrative examples, is easily understood. The more than 700 problems throughout the volume should prove instructive to the student and also to the teacher, by way of problem example. Many objective type problems are included which, Professor Streeter emphasizes in his preface, have the advantage of quickness and ease in correcting.

E. W. Hammer, Jr.

Aerodynamics of Supersonic Flight, by Alan Pope. 184 pages, illustrations, 15 × 23 cm. New York, Pitman Publishing Co., 1950. Price, $4.00.

Professor Pope is to be congratulated on this relatively short but concise coverage of the fundamental aspects of supersonic aerodynamics so sorely neglected in the past. The vast fund of experimental data and the resulting changes in even basic theories have created a need for acquainting the undergraduate student with the general fundamentals of supersonics and their application in a meaningful manner which can provide him with a practical, rather than an abstract and meaningless, background for further study. In addition, this fast-moving field has created many gaps in the tremendous background of knowledge required of the busy practicing aeronautical engineer who has had little or no opportunity to study this specialized subject in college. This excellent book helps to bridge many of these gaps and will provide the too-often unprogressive college curricula with an up-to-date, basic text for an introductory, one-term course at the undergraduate level. Prerequisites for a comprehension of the material presented are just differential and integral calculus and one term of elementary aerodynamics.

Noteworthy are the many illustrative examples showing the application of the theory, and validation of latest theory with experimental results. Also worthy of special mention is a clear-cut analysis of the steps required in "roughing out" the performance of a supersonic ram-jet aircraft and propulsion system. Presentation of the physical interpretations and explanations of supersonic flow phenomena further provide the reader with an effective grasp of the fundamentals.

This book is divided into seven major chapters. Chapter One contains the usual discussion of the fundamental relations. Chapter Two deals with the one-dimensional flow in a duct and is followed by a treatment of the two-dimensional characteristics of supersonic flow in Chapter Three. Chapter Four contains a discussion of the behavior of a supersonic flow at a pronounced concavity or at the forward surface of a wedge, known as the oblique shock. The approximate theories, including their application and limitations, are covered in Chapter Five along with readily usable charts. Chapter Six presents the subject of supersonic wind tunnels—their types, testing techniques, apparatus, and design. Chapter Seven concerns itself with three-dimensional flow and includes a discussion of the induced drag of three-dimensional wings, the types of supersonic wings, the effects of sweepback, the types of wings (rectangular, trapezoidal, sweptback and delta), the limited case of the Busemann biplane wing, axially-symmetrical cones and fuselages, supersonic propulsion systems and finally the analysis of the performance of a supersonic airplane.

SAMUEL M. BERKOWITZ

ELEMENTS OF MODERN PHYSICS, by W. C. Michels and A. L. Patterson. 659 pages, illustrations, 16 × 23 cm. New York, D. Van Nostrand & Co., 1951. Price, $6.50.

Elements is the outgrowth of mimeographed editions used and revised accordingly during the last ten years in a course at Bryn Mawr College which serves at the same time as a first course for physics majors, an allied course for other science majors, and an elective science course for liberal arts students. The book presents an introductory treatment which takes into account the change in thought made necessary by the theoretical developments of the first half of this century, notably the relativity and quantum theories. The prime purpose of the authors throughout the book is to provide a sense of unity and lead the student to the understanding of physics as an integrated whole rather than a collection of studies of distinct and separate phenomena.

To achieve this, topics have been grouped together in each chapter on the basis of similarities in the experimental and theoretical considerations involved. For example, gravitational, electric, and magnetic fields, all of which may be treated by inverse square relationships, are contained in one chapter entitled "Fields of Force." The order of presentation resulting from this grouping of course causes considerable departure from historical order. Some feeling for the chronology of the subject is retained, however, by insertion of the dates of many important discoveries and the scientists responsible for them.

In all, the book contains 24 chapters, the first 16 of which are regarded by the authors as a necessary whole in providing a strong basic background in physics. Of the remaining eight, the choice of the chapters to be covered should be determined by the interests of the instructor and his students and the amount of time remaining in the school year after completion of the 16 basic chapters.

The MKS system of units is used throughout the text in order to simplify the treatment of the subject—use of this system avoids the complication of discussing three separate systems of electrical units while the student is engaged in his initial study of the principles involved. (A thorough discussion of the other systems of units is included in the Appendix.)

Calculus need not be regarded as a prerequisite for a course using this book although these methods are used in the text. The principles of the differential calculus are skillfully introduced in the treatment of linear motions early in the book, and made use of thereafter.

Numerous problems are included which are designed to help the student develop and broaden his method of physical thinking. The answers to the odd-numbered problems are given at the end of the book.

Elements is to be heartily recommended as a *modern* approach to a *modern* subject. The integration of the many ramifications of the ever-broadening field of physics into a comprehensible unity is vitally needed in order that the student may attain a truly strong background in the subject—after ten years of trial and molding, this book emerges as one which goes far toward meeting this need.

JEAN A. MINKIN

BOOK NOTES

THE PREFABRICATION OF HOUSES, by Burnham Kelly. 466 pages, illustrations, 16 × 24 cm. Published jointly by The Technology Press of The Massachusetts Institute of Technology and John Wiley & Sons, Inc., 1951. Price, $7.50.

The study of the prefabrication industry as presented in this volume is extremely broad and would be primarily of interest to those in the building industry. The material is divided into three parts: Part 1 includes a history, a summary of the present state of the industry, and a summary of the ideas concerning possible future of the industry; Part 2 covers the facts concerning design and production processes, management, research and development, procurement, processing and fabrication of materials, and the distribution and marketing of the finished product. Material is so organized that information concerning any one company is to be found under subject subdivisions in which the company is concerned; Part 3 is composed of appendices covering questionaires sent to industry, names of companies, list of prefabricators and a bibliography.

MARINE PRODUCTS OF COMMERCE, by Donald K. Tressler and James McW. Lemon. Second edition, 782 pages, illustrations, 16 × 24 cm. New York, Reinhold Publishing Corp., 1951. Price, $18.00.

A complete, condensed up-to-date treatise on commercial products of the oceans, the technology of obtaining said products and their by-product, their processing, preserving and marketing is presented in this second edition. The coverage is wide and includes details of manufacturing processes, processing machinery, transportation equipment of fresh and frozen fishery products, fishery by-products, commercial uses of algae and sea weed, nutrition factors, gemology and related economic data.

PRINCIPLES OF GENERAL CHEMISTRY, by Stuart R. Brinkley. Fourth edition, 703 pages, tables and diagrams, 16 × 24 cm. New York, The Macmillan Co., 1951. Price, $5.00.

This fourth edition is a bringing up-to-date of the material to be taught in a general inorganic chemistry course at college level. The general order of the principal topics is essentially the same as in the third edition. Chapters dealing with atomic structure, nuclear changes, solutions, ionization and ionic reactions have been reorganized and enlarged. The author has used simple diagrams as illustrative material rather than photographs. The text includes broad coverage of inorganic chemistry and background for qualitative analysis.

AN INTERNATIONAL BIBLIOGRAPHY ON ATOMIC ENERGY, by Atomic Energy Commission Group. Volume 2, SCIENTIFIC ASPECTS. Unpaged, 22 × 28 cm. New York, United Nations Publications, 1951. Price, $10.00 (paper).

Volume 2, *Scientific Aspects*, is an extensive, comprehensive compilation of source material covering nuclear energy and its application as directed toward peaceful purposes in medicine and industry, for the past twenty-five years. The bibliographical material is scientific in nature. The breakdown is by subject with books, articles, etc., listed under those categories to which they are chiefly related, in an alphabetical arrangement. An author index and a listing of abbreviations of the journals used are included.

V. 19

CURRENT TOPICS

New Soldering Process.—A revolutionary new soldering process for use in the manufacture of automotive radiators and other heat exchange equipment has been developed by McCord Corporation of Detroit, Mich., in cooperation with Mathieson Chemical Corporation, Baltimore, Md.

Called the "Coronil Soldering Process," the novel chemical-metallurgical method employs a new series of soldering fluxes bearing the trade name M.C.C. which were discovered by McCord and developed cooperatively with Mathieson. These M.C.C. fluxes are suitable for use in joining most of the commonly used metals and produce solder bonds of highest strength, leave no corrosive residues and eliminate corrosion of plant equipment.

Successful production tests and field studies culminated an intensive research and development program which extended over a period of several years. The new process, considered by McCord as the most important contribution to the art since the development of soldering as a method for joining metals, eliminates substantial corrosion losses previously encountered, simplifies production methods and produces radiators of superior strength which are less likely to be troubled with corrosion in the field.

M.C.C. fluxes are made from hydrazine, a Mathieson chemical with many potential uses. Hydrazine was first used on a large scale as a rocket fuel by the Germans in World War II, and has since found commercial application in the pharmaceutical, agricultural, metal finishing and synthetic fibre fields.

Early research by McCord determined that certain hydrazine compounds answered the requirements of an ideal soldering flux, and a cooperative laboratory and plant experimental program with Mathieson resulted in the development.

The "Coronil Soldering Process," in use for some time at one McCord plant, will be extended to operations in its other plants in the near future. The M.C.C. fluxes, available from Mathieson through an arrangement with McCord, have already found application in the manufacture of electronic equipment and in other industries.

New Type Cathode Gives Electron Tubes Longer Life.—Announcement of a new type cathode by Philips Laboratories, Inc., Irvington-on-Hudson, N.Y., will make possible production of electron tubes of higher current output with longer uniform life. Electron tube design has hitherto been restricted by the limitations of the cathode, electron-emitting heart of every tube and most critical tube element.

According to Dr. O. S. Duffendack, President and Director of Research, "it is the first basically new cathode in over 25 years." Named the "L" Cathode in honor of one of the scientists who originated it in the Philips Research Laboratories in Holland, the cathode was further developed by the Philips Laboratories here.

Entirely new in principle, the cathode from its reservoir produces a regulated amount of emissive material. This provides long uniform tube life with high emission current density. Its emitting surface can be shaped to very close tolerances. The "L" Cathode can do much to improve the performance of electron tubes used in high frequency heating, broadcasting, telephony, calculating machines, instruments, controls and other electronic devices for industrial and military uses.

Leading electronic research laboratories have already used the new cathode with outstanding results in klystrons, disc seal triodes, iconoscopes, magnetrons, special cathode-ray tubes and other types. Experimental samples are now being offered to tube manufacturers.

In its simplest form the "L" cathode consists of two sections formed from one piece of molybdenum. The lower section is open at one end and contains a filament for indirectly heating the cathode. The upper section contains a quantity of the emissive material, a barium compound, sealed by a cap of porous tungsten.

Length of the tube life then depends upon the operating temperature of the tube and the amount of emissive material present which is carried through the porous cap.

Coast-to-Coast Dial Telephone.—The first coast-to-coast dial call by a telephone subscriber will be made this fall, the New Jersey Bell Telephone company has announced.

Equipment required for the initial use of coast-to-coast dialing by customers is being installed in the Englewood exchange and should be in use by the end of the year, the company said.

The intricate switching apparatus, development of which climaxes years of research by the Bell Telephone Laboratories, will enable subscribers to span the continent through 10 pulls of the dial instead of the customary seven now used in completing calls to shorter distances.

In the initial phase of cross-country dialing, 10,000 customers in the Englewood exchange area will be able to dial directly to any of 11,000,000 telephones in such cities as San Francisco, Chicago, Cleveland, Detroit, and Boston and many of their adjacent suburbs, the company said.

"Telephone operators have been dialing calls across the country for several years," the company explained. "At the present time one out of every three long distance calls is dialed directly by the operator but Englewood will mark the first time that subscribers will be able to dial calls covering so wide an area."

"Long distance calls which averaged 14 minutes to connect 30 years ago and which average two minutes today, may be completed in a matter of seconds under subscriber long distance dialing," the company said.

Service improvements have so greatly increased the use of long distance service that twice as many long distance operators are required today as were needed 10 years ago, the company said.

A country-wide numbering plan, which includes parts of Canada, plays a key role in the nationwide dialing system. In all, there are more than 80 numbering plan areas, each of which is assigned a special three digit code.

For example on a call from Englewood to San Francisco, the subscriber will first dial the latter's code of "3–1–8" and follow immediately with the number of the telephone being called. On a call to a subscriber with the number of GArfield 1–9950 in San Francisco, the calling party will dial the 10-digit combination of 318–GA1–9950.

The equipment installed in the Englewood central office stores up the digits dialed and as soon as the 10th digit is recorded, the call is automatically switched over telephone highways to the called telephone in San Francisco. (Calls to some party line telephones may require 11 digits before the call can be completed.)

Similarly a preceding code of "6–1–7" will speed a call to a Boston location.

At the start of the Englewood dial trial, 13 codes will be in use covering areas served by 800 central offices. In addition to the codes for San Francisco and Boston, other areas within the dialing range of Englewood subscribers will be:

Area	Code
Philadelphia	2–1–5
Providence	4–0–1
Pittsburgh	4–1–2
Cleveland	2–1–6
Detroit	3–1–3
Chicago	3–1–2
Milwaukee	4–1–4
Oakland	4–1–5
Sacramento	9–1–6

Additional areas in New York State will also be within reach of Englewood subscribers through the use of two other codes.

The trial customer long distance dialing is limited to Englewood, and the company emphasized that subscribers in the 13 areas will not at this time be able to dial Englewood directly even though they can receive direct dial calls from the Englewood subscribers.

Keeping track of the charges on customer dialed long distance calls will also be done automatically. The company said installations of Automatic Message Accounting equipment (abbreviated AMA) will register each step of a call, including the number of the calling telephone, the exchange and number of the called telephone, date and time of day and length of call.

The information is recorded on paper tape which is later run through a decoding machine which assembles, translates, sorts, summarizes and prints information in a form that can be used in preparing customers' bills. Busy and don't answer calls are automatically eliminated by the AMA machinery.

The first coast-to-coast dial call, the company pointed out, comes at a time when the world is observing the 75th anniversary of the invention of the telephone by Alexander Graham Bell and 36 years after the first coast-to-coast telephone call in 1915.

Compact, Inexpensive, New Electronic Brain.—A new electronic "brain" which sells for less than $8000 and which takes up no more space than a 4-drawer filing cabinet is now available. This new Brain, more accurately termed an electronic analog computer and nicknamed "Ida" (for Integro-Differential Analyzer), has been placed on the market by the Computer Corporation of America, New York City.

Especially useful in engineering design, electronic analog computers eliminate the necessity of building costly experimental prototype models. The flight of an aeroplane built with a certain set of aerodynamic characteristics, for example, can be accurately simulated in the computer so that the plane can be "flight tested" before it leaves the drawing boards. In the same way similar dynamic situations can be reproduced and the behavior of various engineering factors in those situations can be determined on the computer.

A computer reduces months of calculation time to minutes.

At a special press demonstration, it was shown how "Ida" can be applied to problems of aerodynamic design. Every ten seconds a "plane" with a different set of design characteristics was "built" and flight tested until the one which behaved best in the "air" was determined.

"Ida's low cost will make it possible for small businesses, research laboratories, universities and engineering consultants to acquire an analog computer with all the time-saving, labor-reducing possibilities that go with it," Mr. Seymour Bosworth, Vice President & Sales Manager of the computer firm said. "Organizations which already have differential analyzers will now find it possible to expand their facilities with a small expenditure," he added.

Previous "brains," according to Mr. Bosworth, have been so large and costly as to be available to but a limited number of organizations, in many cases requiring government financing. "Ida" should do for electronic computation what the mass-produced Ford did for auto transportation, he said.

Types of problems "Ida" can solve, Mr. Bosworth explained, include qualitative and quantitative dynamic studies in chemistry, mechanical engineering, aerodynamics, nucleonics, thermodynamics, servo-analysis and net work analysis. It solves linear differential equations with constant co-efficients up to the eighth order. Eight initial conditions may be set into the problem.

Auxiliary equipment extends "Ida's" usefulness to the solution of differential equations of higher order than eight, differential equations with non-constant co-efficients and non-linear differential equations.

Problems can be worked on either a real or modified time base.

Solutions are recorded in graphic form on a two-channel oscillograph. All computed quantities can be recorded, two at a time. Any computed quantity can also be viewed on an oscilloscope while being recorded.

According to the computer firm's engineers, "Ida" was designed with two main considerations, reasonable cost and simplicity of operation.

Low cost was achieved by the elimination of frills and reduction in the number of parts used, they said. They hope to realize additional economies through mass-production of the unit.

They pointed to these innovations as contributing to simplicity of operation:

The use of patchcords, common in previous and larger computers, has been eliminated. Problems are prepared on a set-up board, which, when in place in the machine, automatically makes all necessary electrical connections.

Problems in the machine can be changed in the few seconds it takes to slide out one set-up board and slide in another. The problems can be stored on the set-up boards for future use.

A minimum of controls are used. Two control knobs apply the forcing function and start and stop the computing process.

Simplicity of maintenance has been achieved, they said, by use of standardized, interchangeable parts of which the amplifiers are typical. All 20 amplifiers are identical, interchangeable, plug-in units.

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19